Maximum A Posteriori (MAP) Estimation

Pieter Abbeel
UC Berkeley EECS
Overview

- **Filtering:**
  \[ P(x_t | z_{0:t}) \]

- **Smoothing:**
  \[ P(x_t | z_{0:T}) \]

- **MAP:**
  \[ \max_{x_{0:T}} P(x_{0:T} | z_{0:T}) \]
MAP

\[
\begin{align*}
\max_{x_0, x_1, x_2, x_3} & \quad P(x_0, x_1, x_2, x_3 | z_0, z_1, z_2, z_3) \\
\propto & \quad \max_{x_0, x_1, x_2, x_3} P(x_0, x_1, x_2, x_3, z_0, z_1, z_2, z_3) \\
= & \quad \max_{x_0, x_1, x_2, x_3} P(z_3 | x_3) P(x_3 | x_2) P(z_2 | x_2) P(x_2 | x_1) P(z_1 | x_1) P(x_1 | x_0) P(z_0 | x_0) P(x_0) \\
= & \quad \max_{x_3} \left( P(z_3 | x_3) \max_{x_2} \left( P(x_3 | x_2) P(z_2 | x_2) \max_{x_1} \left( P(x_2 | x_1) P(z_1 | x_1) \max_{x_0} \left( P(x_1 | x_0) P(z_0 | x_0) P(x_0) \right) \right) \right) \right) \\
& \quad m_0(x_0) \\
m_1(x_1) \\
m_2(x_2) \\
m_3(x_3)
\end{align*}
\]

Generally:

\[
\begin{align*}
m_t(x_t) & = \max_{x_{0:t-1}} P(x_{0:t}, z_{0:t}) \\
& \quad = \max_{x_{0:t-1}} P(x_t | x_{t-1}) P(z_t | x_t) P(x_{0:t-1}, z_{0:t-1}) \\
& \quad = P(z_t | x_t) \max_{x_{t-1}} P(x_t | x_{t-1}) \max_{x_{0:t-2}} P(x_{0:t-1}, z_{0:t-1}) \\
& \quad = P(z_t | x_t) \max_{x_{t-1}} P(x_t | x_{t-1}) m_{t-1}(x_{t-1})
\end{align*}
\]
MAP --- Complete Algorithm

1. Init: $m_0(x_0) = P(z_0|x_0)P(x_0)$

2. For all $t = 1, 2, \ldots, T - 1$
   - For all $x_t$: $m_t(x_t) = P(z_t|x_t) \max_{x_{t-1}} P(x_t|x_{t-1})m_{t-1}(x_{t-1})$
   - For all $x_t$: Store argmax in $\text{pointer}_{t \to t-1}(x_t)$

3. maximum = $\max_{x_T} m_T(x_T)$

4. $x_T^* = \arg \max_{x_T} m_T(x_T)$

5. For all $t = T, T - 1, \ldots, 1$
   - $x_{t-1}^* = \text{pointer}_{t \to t-1}(x_t^*)$

- $O(T \, n^2)$
Summations $\rightarrow$ integrals

But: can’t enumerate over all instantiations

However, we can still find solution efficiently:

- the joint conditional $P(x_{0:T} \mid z_{0:T})$ is a multivariate Gaussian
- for a multivariate Gaussian the most likely instantiation equals the mean

$\rightarrow$ we just need to find the mean of $P(x_{0:T} \mid z_{0:T})$

- the marginal conditionals $P(x_t \mid z_{0:T})$ are Gaussians with mean equal to the mean of $x_t$ under the joint conditional, so it suffices to find all marginal conditionals
- We already know how to do so: marginal conditionals can be computed by running the Kalman smoother.

Alternatively: solve convex optimization problem