Non-Convex Optimization through Sequential Convex Programming

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Non-Convex Optimization

- Reminder: Convex optimization:

$$\begin{aligned} \min_x \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0 \quad \forall i \\ & A(j,:)x - b_j = 0 \quad \forall j \end{aligned}$$

with f_i convex

- Non-convex optimization:

$$\begin{aligned} \min_x \quad & g_0(x) \\ \text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i \\ & h_j(x) = 0 \quad \forall j \end{aligned}$$

with:

g_i non-convex

h_j nonlinear

Sequential Convex Programming

- *To solve:*
$$\begin{aligned} \min_x \quad & g_0(x) \\ \text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i \\ & h_j(x) = 0 \quad \forall j \end{aligned} \qquad (1)$$

merit function

- *Solve:*
$$\min_x \quad g_0(x) + \mu \sum_i |g_i(x)|^+ + \mu \sum_j |h_j(x)| \quad = \min_x \quad f_\mu(x) \qquad (2)$$

and increase μ in an outer loop until the two sums equal zero.

- *To solve (2), repeatedly solve the convex program:*

$$\begin{aligned} \min_x \quad & g_0(\bar{x}) + \nabla_x g_0(\bar{x})(x - \bar{x}) \\ & + \mu \sum_i |g_i(\bar{x}) + \nabla_x g_i(\bar{x})(x - \bar{x})|^+ \\ & + \mu \sum_j |h_j(\bar{x}) + \nabla_x h_j(\bar{x})(x - \bar{x})| \\ \text{s.t.} \quad & \|x - \bar{x}\|_2 \leq \varepsilon \end{aligned}$$

$\bar{x}$: current point

(trust region constraint)

Sequential Convex Programming

Inputs: $\bar{x}, \mu = 1, \varepsilon_0, \alpha \in (0.5, 1), \beta \in (0, 1), t \in (1, \infty)$

While ($\sum_i |g_i(\bar{x})|^+ + \sum_j |h_j(\bar{x})| \geq \delta$ AND $\mu < \mu_{\mathrm{MAX}}$)

$\mu \leftarrow t\mu, \quad \varepsilon \leftarrow \varepsilon_0$ // increase penalty coefficient for constraints; re-init trust region size

While (1) // [2] loop that optimizes f_μ

Compute terms of first-order approximations: $g_0(\bar{x}), \nabla_x g_0(\bar{x}), g_i(\bar{x}), \nabla_x g_i(\bar{x}), h_j(\bar{x}), \nabla_x h_j(\bar{x}), \quad \forall i, j$

While (1) // [3] loop that does trust-region size search

Call convex program solver to solve:

$$\begin{aligned}\left(\bar{f}_\mu(\bar{x}_{\mathrm{next?}}), \bar{x}_{\mathrm{next?}}\right) = \min_x \quad & g_0(\bar{x}) + \nabla_x g_0(\bar{x})(x - \bar{x}) \\ & + \mu \sum_i |g_i(\bar{x}) + \nabla_x g_i(\bar{x})(x - \bar{x})|^+ + \mu \sum_j |h_j(\bar{x}) + \nabla_x h_j(\bar{x})(x - \bar{x})| \\ \text{s.t.} \quad & \|x - \bar{x}\|_2 \leq \varepsilon\end{aligned}$$

If $\dfrac{f_\mu(\bar{x}) - f_\mu(\bar{x}_{\mathrm{next?}})}{\bar{f}_\mu(\bar{x}) - \bar{f}_\mu(\bar{x}_{\mathrm{next?}})} \geq \alpha$

Then shrink trust region: $\varepsilon \leftarrow \beta\varepsilon$

Else Update $\bar{x} \leftarrow \bar{x}_{\mathrm{next?}}$, Grow trust region: $\varepsilon \leftarrow \varepsilon/\beta$, and Break out of while [3]

If ε below some threshold, Break out of while [2] and while [3]

Non-Convex Optimization

- Non-convex optimization with convex parts separated:

$$\begin{aligned} \min_x \quad & f_0(x) + g_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0 \quad \forall i \\ & Ax - b = 0 \quad \forall j \\ & g_k(x) \leq 0 \quad \forall k \\ & h_l(x) = 0 \quad \forall l \end{aligned}$$

with:
f_i convex
g_k non-convex
h_l nonlinear

- Retain convex parts and in inner loop solve:

$$\begin{aligned} \min_x \quad & f_0(x) + g_0(x) + \mu \sum_k |g_k(x)|^+ + \mu \sum_l |h_l(x)| \\ \text{s.t.} \quad & f_i(x) \leq 0 \quad \forall i \\ & Ax - b = 0 \quad \forall j \end{aligned}$$