Non-Convex Optimization through Sequential Convex Programming

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Non-Convex Optimization

- Reminder: Convex optimization:

\[
\min_{x} \quad f_0(x) \\
\text{s.t.} \quad f_i(x) \leq 0 \quad \forall i \\
\quad \quad A(j,:)x - b_j = 0 \quad \forall j
\]

- Non-convex optimization:

\[
\min_{x} \quad g_0(x) \\
\text{s.t.} \quad g_i(x) \leq 0 \quad \forall i \\
\quad \quad h_j(x) = 0 \quad \forall j
\]

with \( f_i \) convex

with:

- \( g_i \) non-convex
- \( h_j \) nonlinear
Sequential Convex Programming

- **To solve:** \[ \min_x g_0(x) \]
  \[ \text{s.t. } g_i(x) \leq 0 \quad \forall i \]
  \[ h_j(x) = 0 \quad \forall j \]

- **Solve:** \[ \min_x g_0(x) + \mu \sum_i |g_i(x)|^+ + \mu \sum_j |h_j(x)| = \min_x f_\mu(x) \] (2)

and increase \( \mu \) in an outer loop until the two sums equal zero.

- **To solve (2), repeatedly solve the convex program:**

  \[ \min_x g_0(\bar{x}) + \nabla_x g_0(\bar{x})(x - \bar{x}) \]
  \[ + \mu \sum_i |g_i(\bar{x}) + \nabla_x g_i(\bar{x})(x - \bar{x})|^+ \]
  \[ + \mu \sum_j |h_j(\bar{x}) + \nabla_x h_j(\bar{x})(x - \bar{x})| \]

  \[ \text{s.t. } \|x - \bar{x}\|_2 \leq \varepsilon \]

  (trust region constraint)
Inputs: $\bar{x}, \mu = 1, \varepsilon_0, \alpha \in (0.5, 1), \beta \in (0, 1), t \in (1, \infty)$

While $(\sum_i |g_i(\bar{x})|^+ + \sum_j |h_j(\bar{x})| \geq \delta \land \mu < \mu_{\text{MAX}})$

$\mu \leftarrow t \mu$, $\varepsilon \leftarrow \varepsilon_0$  // increase penalty coefficient for constraints; re-init trust region size

While (1)  // [2] loop that optimizes $f_\mu$

Compute terms of first-order approximations: $g_0(\bar{x}), \nabla_x g_0(\bar{x}), g_i(\bar{x}), \nabla_x g_i(\bar{x}), h_j(\bar{x}), \nabla_x h_j(\bar{x}), \forall i, j$

While (1)  // [3] loop that does trust-region size search

Call convex program solver to solve:

$$(\bar{f}_\mu(\bar{x}_{\text{next}}), \bar{x}_{\text{next}}) = \min_x \ g_0(\bar{x}) + \nabla_x g_0(\bar{x})(x - \bar{x})$$
$$+ \mu \sum_i |g_i(\bar{x}) + \nabla_x g_i(\bar{x})(x - \bar{x})|^+ + \mu \sum_j |h_j(\bar{x}) + \nabla_x h_j(\bar{x})(x - \bar{x})|$$

s.t. $\|x - \bar{x}\|_2 \leq \varepsilon$

If $\frac{f_\mu(\bar{x}) - f_\mu(\bar{x}_{\text{next}})}{f_\mu(\bar{x}) - f_\mu(\bar{x}_{\text{next}})} \geq \alpha$

Then shrink trust region: $\varepsilon \leftarrow \beta \varepsilon$

Else Update $\bar{x} \leftarrow \bar{x}_{\text{next}}$, Grow trust region: $\varepsilon \leftarrow \varepsilon / \beta$, and Break out of while [3]

If $\varepsilon$ below some threshold, Break out of while [3] and while [2]
Non-Convex Optimization

- Non-convex optimization with convex parts separated:

\[
\begin{align*}
\min_x & \quad f_0(x) + g_0(x) \\
\text{s.t.} & \quad f_i(x) \leq 0 \quad \forall i \\
& \quad Ax - b = 0 \quad \forall j \\
& \quad g_k(x) \leq 0 \quad \forall k \\
& \quad h_l(x) = 0 \quad \forall l
\end{align*}
\]

with:
- \( f_i \) convex
- \( g_k \) non-convex
- \( h_l \) nonlinear

- Retain convex parts and in inner loop solve:

\[
\begin{align*}
\min_x & \quad f_0(x) + g_0(x) + \mu \sum_k |g_k(x)|^+ + \mu \sum_l |h_l(x)| \\
\text{s.t.} & \quad f_i(x) \leq 0 \quad \forall i \\
& \quad Ax - b = 0 \quad \forall j
\end{align*}
\]