Smoother

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Many slides adapted from Thrun, Burgard and Fox, Probabilistic Robotics

Overview

- **Filtering:**
  \[ P(x_t | z_0, z_1, \ldots, z_t) \]

- **Smoothing:**
  \[ P(x_t | z_0, z_1, \ldots, z_T) \]

  Note: by now it should be clear that the “u” variables don’t really change anything conceptually, and going to leave them out to have less symbols appear in our equations.
Generally, recursively compute:

\[
P(x_{t+1}, z_0, \ldots, z_t) = \sum_{x_t} P(x_{t+1} | x_t) P(x_t, z_0, \ldots, z_t)
\]

\[
P(x_{t+1}, z_0, \ldots, \tilde{z}_{t+1}) = p(\tilde{z}_{t+1} | x_{t+1}) P(x_{t+1}, z_0, \ldots, z_t)
\]
Complete Smoother Algorithm

- **Forward pass (= filter):**
  1. Init: $a_0(x_0) = P(z_0|x_0)P(x_0)$
  2. For $t = 0, \ldots, T - 1$
     - $a_{t+1}(x_{t+1}) = P(z_{t+1}|x_{t+1}) \sum_{x_t} P(x_{t+1}|x_t)a_t(x_t)$

- **Backward pass:**
  1. Init: $b_T(x_T) = 1$
  2. For $t = T - 1, \ldots, 0$
     - $b_t(x_t) = \sum_{x_{t+1}} P(x_{t+1}|x_t)P(z_{t+1}|x_{t+1})b_{t+1}(x_{t+1})$

- **Combine:**
  1. For $t = 0, \ldots, T$
     - $P(x_t, z_0, \ldots, z_T) = a_t(x_t)b_t(x_t)$

Note 1: computes for all times $t$ in one forward+backward pass
Note 2: can find $P(x_t | z_0, \ldots, z_T)$ by simply renormalizing

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Important Variation

- **Find** $P(x_t, x_{t+1}, z_0, \ldots, z_T)$

- **Recall:**
  - $a_t(x_t) = P(x_t, z_0, \ldots, z_T)$
  - $b_t(x_t) = P(z_{t+1}, \ldots, z_T | x_t)$

- **So we can readily compute**

$$P(x_t, x_{t+1}, z_0, \ldots, z_T)$$
$$= P(x_t, z_0, \ldots, z_T)P(x_{t+1}|x_t, z_0, \ldots, z_T)P(z_{t+1}, \ldots, z_T | x_{t+1}, x_t, z_0, \ldots, z_T)$$
$$= P(x_t, z_0, \ldots, z_t)P(a_{t+1}|x_t)P(z_{t+1}, \ldots, z_T | x_{t+1})$$
$$= a_t(x_t)P(x_{t+1}|x_t)b_{t+1}(x_{t+1})$$

(Law of total probability)
(Markov assumptions)
(definitions a, b)
Exercise

- Find \( P(x_t, x_{t+k}, z_0, \ldots, z_T) \)

Kalman Smoother

- = smoother we just covered instantiated for the particular case when \( P(x_{t+1} \mid X_t) \) and \( P(z_t \mid X_t) \) are linear Gaussians

- We already know how to compute the forward pass (=Kalman filtering)

- Backward pass:

\[
    b_t(x_t) = \int_{x_{t+1}} P(x_{t+1} \mid x_t) P(z_{t+1} \mid x_{t+1}) b_{t+1}(x_{t+1}) dx_{t+1}
\]

- Combination:

\[
    P(x_t, z_0, \ldots, z_T) = a_t(x_t) b_t(x_t)
\]
Kalman Smoother Backward Pass

- TODO: work out integral for $b_t$
- TODO: insert backward pass update equations
- TODO: insert combination $\rightarrow$ bring renormalization constant up front so it's easy to read off $P(X_t | Z_0, \ldots, Z_T)$

Matlab code data generation example

- $A = \begin{bmatrix} 0.99 & 0.0074 \\ -0.0136 & 0.99 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$
- $x(:,1) = [-3;2]$;
- $\Sigma_w = \text{diag}(\begin{bmatrix} 0.3 & 0.7 \end{bmatrix})$; $\Sigma_v = \begin{bmatrix} 2 & 0.05 \\ 0.05 & 1.5 \end{bmatrix}$
- $w = \text{randn}(2,T); w = \text{sqrtm}(\Sigma_w)w'; v = \text{randn}(2,T); v = \text{sqrtm}(\Sigma_v)v'$
- for $t=1:T-1$
  - $x(:,t+1) = A \cdot x(:,t) + w(:,t);$ 
  - $y(:,t) = C \cdot x(:,t) + v(:,t);$ 
  - end
- % now recover the state from the measurements
- $P_0 = \text{diag}([100 100]); x0 = [0; 0];$
- % run Kalman filter and smoother here
- % + plot
Kalman filter/smoothing example