**CS 287: Advanced Robotics**
**Fall 2009**

Lecture 3: Control 2: Fully actuated wrap-up/recap, Lyapunov direct method, Optimal control

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## Fully actuated recap


$\ddot{q} = f_1(q, \dot{q}, t) + f_2(q, \dot{q}, t)u$ fully actuated in $(q, \dot{q}, t)$ iff $\text{rank} f_2(q, \dot{q}, t) = \dim q$.

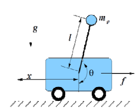
- Given a ***fully actuated system*** and a (smooth) target trajectory $q^*$
  - Can solve dynamics equations for required control inputs = "feedforward controls"
    $$u_{\text{ff}}(t) = f_2^{-1}(q^*, \dot{q}^*, t)\,(\ddot{q}^* - f_1(q^*, \dot{q}^*, t))$$
  - Feedforward control is insufficient in presence of
    - Model inaccuracy
    - Perturbations + instability
  - Proportional feedback control can alleviate some of the above issues.
    - Steady state error reduced by (roughly) factor $K_p$, but large $K_p$ can be problematic in presence of delay → Add integral term
    - Ignores momentum → Add derivative term

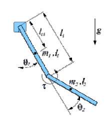
$$u(t) = u_{\text{ff}}(t) + K_p(q^*(t) - q(t)) + K_d(\dot{q}^*(t) - \dot{q}(t)) + K_i \int_0^t (q^*(\tau) - q(\tau))d\tau$$

  - PID constants require tuning: often by hand, Ziegler-Nichols and TLC provide good starting points, (policy search could automate this)
  - If control inputs do not directly relate to the degrees of freedom, we can use feedback linearization to get that form:
    $$\ddot{q}(t) = v(t) \qquad u(t) = f_2^{-1}(q, \dot{q}, t)\,(v(t) - f_1(q, \dot{q}, t))$$

## Today's lecture

- Fully actuated recap [done]
- Aside on integrator wind-up
- Lyapunov direct method --- a method that can be helpful in proving guarantees about controllers
- Optimal control





## Readings for today's lecture

- --
- Optional:
  - Tedrake Chapter 3 [optional, nice read on energy pumping control strategies]
  - Slotine and Li, Example 3.21, Global asymptotic stability of a robot position controller [optional]

## Aside: Integrator wind-up

$$u(t) = u_{\text{ff}}(t) + K_p(q^*(t) - q(t)) + K_d(\dot{q}^*(t) - \dot{q}(t)) + K_i \int_0^t (q^*(\tau) - q(\tau))d\tau$$

- Recipe: Stop integrating error when the controls saturate
- Reason: Otherwise it will take a long time to react in the opposite direction in the future.
- Matters in practice!

[See also Astrom and Murray, Section 10.4]

## Lyapunov

- Lyapunov theory is used to make conclusions about trajectories of a system without finding the trajectories (i.e., solving the differential equation)
- A typical Lyapunov theorem has the form:
  - if there exists a function $V: R^n \rightarrow R$ that satisfies some conditions on $V$ and \dot{V}
  - then, trajectories of system satisfy some property
- If such a function V exists we call it a Lyapunov function
- Lyapunov function V can be thought of as generalized energy function for system

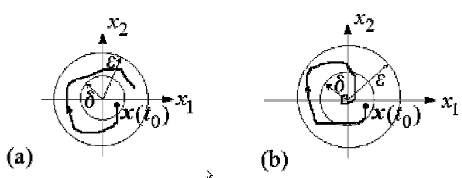
## Guarantees?

**Equilibrium state.** A state $x^*$ is an equilibrium state of the system $\dot{x} = f(x)$ if $f(x^*) = 0$.

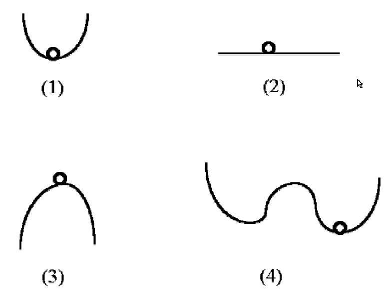
**Stability.** The equilibrium state $x^*$ is said to be stable if, for any $R > 0$, there exists $r > 0$, such that if $\|x(0) - x^*\| < r$, then $\|x(t) - x^*\| < R$ for all $t \geq 0$. Otherwise, the equilibrium point is unstable.

**Asymptotic stability.** An equilibrium point $x^*$ is asymptotically stable if it is stable, and if in addition there exists some $r > 0$ such that $\|x(0) - x^*\| < r$ implies that $x(t) \rightarrow x^*$ as $t \rightarrow \infty$.

## Stability illustration




## Simple physical examples



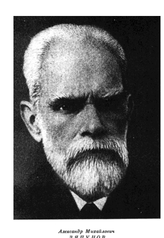

## Proving stability

- To prove stability, we need to show something about the solution of a non-linear differential equation for all initial conditions within a certain radius of the equilibrium point.
- Challenge: typically no closed form solution to differential equation!

→ How to analyze / prove stability ??

## Alexandr Mikhailovich Lyapunov

- In late 19th century introduced one of the most useful and general approaches for studying stability of non-linear systems.
- [Lyapunov's PhD thesis: 1892]



## A Lyapunov boundedness theorem

suppose there is a function $V$ that satisfies

- all sublevel sets of $V$ are bounded
- $\dot{V}(z) \le 0$ for all $z$

then, all trajectories are bounded, *i.e.*, for each trajectory $x$ there is an $R$ such that $\|x(t)\| \le R$ for all $t \ge 0$

in this case, $V$ is called a Lyapunov function (for the system) that proves the trajectories are bounded

[from Boyd, ee363]

## Proof:

to prove it, we note that for any trajectory $x$

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))\, d\tau \le V(x(0))$$

so the whole trajectory lies in $\{z \mid V(z) \le V(x(0))\}$, which is bounded

also shows: every sublevel set $\{z \mid V(z) \le a\}$ is invariant

[from Boyd, ee363]

## A Lyapunov global asymptotic stability theorem

suppose there is a function $V$ such that

- $V$ is positive definite
- $\dot{V}(z) < 0$ for all $z \ne 0$, $\dot{V}(0) = 0$

then, every trajectory of $\dot{x} = f(x)$ converges to zero as $t \to \infty$ (*i.e.*, the system is globally asymptotically stable)

intepretation:

- $V$ is positive definite generalized energy function
- energy is always dissipated, except at $0$

[from Boyd, ee363]

## Proof:

Suppose trajectory $x(t)$ does not converge to zero.

$V(x(t))$ is decreasing and nonnegative, so it converges to, say, $\epsilon$ as $t \to \infty$.

Since $x(t)$ does not converge to 0, we must have $\epsilon > 0$, so for all $t$, $\epsilon \le V(x(t)) \le V(x(0))$.

$C = \{z | \epsilon \le V(z) \le V(x(0))\}$ is closed and bounded, hence compact. So $\dot{V}$ (assumed continuous) attains its supremum on $C$, i.e., $\sup_{z \in C} \dot{V} = -a < 0$. Since $\dot{V}(x(t)) \le -a$ for all $t$, we have

$$V(x(T)) = V(x(0)) + \int_0^T \dot{V}(x(t))dt \le V(x(0)) - aT$$

which for $T > V(x(0))/a$ implies $V(x(T)) < 0$, a contradiction.

So every trajectory $x(t)$ converges to 0, i.e., $\dot{x} = f(x)$ is globally asymptotically stable.

[from Boyd, ee363]

**Global invariant set Theorem.** Assume that

- $V(x) \to \infty$ as $\|x\| \to \infty$.
- $\dot{V}(x) \le 0$ over the whole state space.

Let $R$ be the set of all points where $\dot{V(x)} = 0$, and let $M$ be the largest invariant set in $R$. Then all solutions globally asymptotically converge to $M$ as $t \to \infty$.

## Example 1

$$\ddot{q} + \dot{q} + \sin q = u$$

$$u = \sin q + K_p(q^* - q) + K_d(0 - \dot{q})$$

## Example 1 (solution)

$$\ddot{q} + \dot{q} + g(q) = u$$
$$u = g(q) + K_d(q^* - q) + K_d(0 - \dot{q})$$

We choose $V = \frac{1}{2}K_p(q - q^*)^2 + \frac{1}{2}\dot{q}^2$.
This gives for $\dot{V}$:

$$\begin{aligned} \dot{V} &= K_p(q - q^*)\dot{q} + \dot{q}\ddot{q} \\ &= K_p(q - q^*)\dot{q} + \dot{q}\left(K_p(q^* - q) - K_d\dot{q} - \dot{q}\right) \\ &= -(1 + K_d)\dot{q} \end{aligned}$$

Hence $V$ satisfies: (i) $V(q) \geq 0$ and $= 0$ iff $q = q^*$, (ii) $\dot{V} \leq 0$. Since the arm cannot get "stuck" at any position such that $q \neq 0$ (which can be easily shown by noting that acceleration is non-zero in such situations), the robot arm must settle down at $\dot{q} = 0$ and $q = 0$, according to the invariant set theorem. Thus the system is globally asymptotically stable.

## Example 2: PD controllers are stable for fully actuated manipulators

- (Slotine and Li, Example 3.21.)

### A converse Lyapunov G.E.S. theorem

suppose there is $\beta > 0$ and $M$ such that each trajectory of $\dot{x} = f(x)$ satisfies

$$\|x(t)\| \leq Me^{-\beta t}\|x(0)\| \text{ for all } t \geq 0$$

(called *global exponential stability*, and is stronger than G.A.S.)

then, there is a Lyapunov function that proves the system is exponentially stable, *i.e.*, there is a function $V : \mathbf{R}^n \to \mathbf{R}$ and constant $\alpha > 0$ s.t.

- $V$ is positive definite
- $\dot{V}(z) \leq -\alpha V(z)$ for all $z$

[from Boyd, ee363]

### Proof of converse G.E.S. Lyapunov theorem

suppose the hypotheses hold, and define

$$V(z) = \int_0^\infty \|x(t)\|^2 \, dt$$

where $x(0) = z$, $\dot{x} = f(x)$

since $\|x(t)\| \leq Me^{-\beta t}\|z\|$, we have

$$V(z) = \int_0^\infty \|x(t)\|^2 \, dt \leq \int_0^\infty M^2 e^{-2\beta t}\|z\|^2 \, dt = \frac{M^2}{2\beta}\|z\|^2$$

(which shows integral is finite)

[from Boyd, ee363]

### Finding Lyapunov functions

- there are many different types of Lyapunov theorems
- the key in all cases is to *find* a Lyapunov function and verify that it has the required properties
- there are several approaches to finding Lyapunov functions and verifying the properties

one common approach:

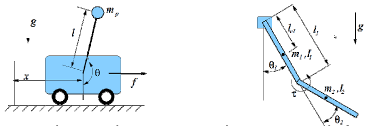
- decide form of Lyapunov function (*e.g.*, quadratic), parametrized by some parameters (called a *Lyapunov function candidate*)
- try to find values of parameters so that the required hypotheses hold

[from Boyd, ee363]

## Lyapunov recap

- Enables providing stability guarantees w/o solving the differential equations for all possible initial conditions!
- Tricky part: finding a Lyapunov function
- A lot more to it than we can cover in ½ lecture, but this should provide you with a starting point whenever you might need something like this in the future

# Intermezzo on energy pumping



- An interesting and representative approach from non-linear control:
  - Energy pumping to swing up:
    - Write out energy of system E(q, \dot{q})
    - Write out time derivative of energy: \{dot}E (q, \dot{q}, u)
    - Choose u as a function of q and \dot{q} such that energy is steered towards required energy to reach the top
  - Local controller to stabilize at top --- we will see this aspect in detail later.

*Energy pumping nicely described in Tedrake Chapter 3. Enjoy the optional read!*

# Forthcoming lectures

- Optimal control: provides general computational approach to tackle control problems---both under- and fully actuated.
  - Dynamic programming
    - Discretization
  - Dynamic programming for linear systems
    - Extensions to nonlinear settings:
      - Local linearization
      - Differential dynamic programming
      - Feedback linearization
  - Model predictive control (MPC)
  - Examples: