

CS 287: Advanced Robotics

Fall 2009

Lecture 3: Control 2: Fully actuated wrap-up/recap, Lyapunov direct method, Optimal control

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Fully actuated recap



$\ddot{q} = f_1(q, \dot{q}, t) + f_2(q, \dot{q}, t)u$ fully actuated in (q, \dot{q}, t) iff $\text{rank} f_2(q, \dot{q}, t) = \text{dim} q$.

- Given a **fully actuated system** and a (smooth) target trajectory q^*
 - Can solve dynamics equations for required control inputs = "feedforward controls"

$$u_{\text{ff}}(t) = f_2^{-1}(q^*, \dot{q}^*, t) (\ddot{q}^* - f_1(q^*, \dot{q}^*, t))$$

- Feedforward control is insufficient in presence of
 - Model inaccuracy
 - Perturbations + instability
- Proportional feedback control can alleviate some of the above issues.
 - Steady state error reduced by (roughly) factor K_p , but large K_p can be problematic in presence of delay → Add integral term
 - Ignores momentum → Add derivative term

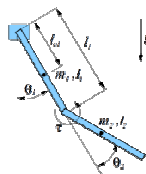
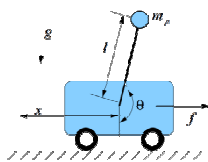
$$u(t) = u_{\text{ff}}(t) + K_p(q^*(t) - q(t)) + K_d(\dot{q}^*(t) - \dot{q}(t)) + K_i \int_0^t (q^*(\tau) - q(\tau)) d\tau$$

- PID constants require tuning: often by hand, Ziegler-Nichols and TLC provide good starting points, (policy search could automate this)
- If control inputs do not directly relate to the degrees of freedom, we can use feedback linearization to get that form:

$$\ddot{q}(t) = v(t) \quad u(t) = f_2^{-1}(q, \dot{q}, t) (v(t) - f_1(q, \dot{q}, t))$$

Today's lecture

- Fully actuated recap [done]
- Aside on integrator wind-up
- Lyapunov direct method --- a method that can be helpful in proving guarantees about controllers
- Optimal control



Readings for today's lecture

- --
- Optional:
 - Tedrake Chapter 3 [optional, nice read on energy pumping control strategies]
 - Slotine and Li, Example 3.21, Global asymptotic stability of a robot position controller [optional]

Aside: Integrator wind-up

$$u(t) = u_{ff}(t) + K_p(q^*(t) - q(t)) + K_d(\dot{q}^*(t) - \dot{q}(t)) + K_i \int_0^t (q^*(\tau) - q(\tau)) d\tau$$

- Recipe: Stop integrating error when the controls saturate
- Reason: Otherwise it will take a long time to react in the opposite direction in the future.
- Matters in practice!

[See also Astrom and Murray, Section 10.4]

Lyapunov

- Lyapunov theory is used to make conclusions about trajectories of a system without finding the trajectories (i.e., solving the differential equation)
- A typical Lyapunov theorem has the form:
 - if there exists a function $V: R^n \rightarrow R$ that satisfies some conditions on V and \dot{V}
 - then, trajectories of system satisfy some property
- If such a function V exists we call it a Lyapunov function
- Lyapunov function V can be thought of as generalized energy function for system

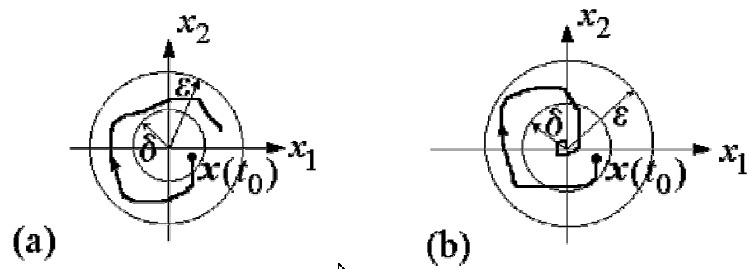
Guarantees?

Equilibrium state. A state x^* is an equilibrium state of the system $\dot{x} = f(x)$ if $f(x^*) = 0$.

Stability. The equilibrium state x^* is said to be stable if, for any $R > 0$, there exists $r > 0$, such that if $\|x(0) - x^*\| < r$, then $\|x(t) - x^*\| < R$ for all $t \geq 0$. Otherwise, the equilibrium point is unstable.

Asymptotic stability. An equilibrium point x^* is asymptotically stable if it is stable, and if in addition there exists some $r > 0$ such that $\|x(0) - x^*\| < r$ implies that $x(t) \rightarrow x^*$ as $t \rightarrow \infty$.

Stability illustration



Simple physical examples



(1)



(2)



(3)



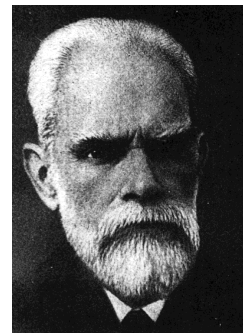
(4)

Proving stability

- To prove stability, we need to show something about the solution of a non-linear differential equation for all initial conditions within a certain radius of the equilibrium point.
 - Challenge: typically no closed form solution to differential equation!
- How to analyze / prove stability ??

Alexandr Mikhailovich Lyapunov

- In late 19th century introduced one of the most useful and general approaches for studying stability of non-linear systems.
- [Lyapunov's PhD thesis: 1892]



*Александр Михайлович
ЛЯПУНОВ*

A Lyapunov boundedness theorem

suppose there is a function V that satisfies

- all sublevel sets of V are bounded
- $\dot{V}(z) \leq 0$ for all z

then, all trajectories are bounded, *i.e.*, for each trajectory x there is an R such that $\|x(t)\| \leq R$ for all $t \geq 0$

in this case, V is called a Lyapunov function (for the system) that proves the trajectories are bounded

[from Boyd, ee363]

Proof:

to prove it, we note that for any trajectory x

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau)) d\tau \leq V(x(0))$$

so the whole trajectory lies in $\{z \mid V(z) \leq V(x(0))\}$, which is bounded

also shows: every sublevel set $\{z \mid V(z) \leq \alpha\}$ is invariant

[from Boyd, ee363]

A Lyapunov global asymptotic stability theorem

suppose there is a function V such that

- V is positive definite
- $\dot{V}(z) < 0$ for all $z \neq 0$, $\dot{V}(0) = 0$

then, every trajectory of $\dot{x} = f(x)$ converges to zero as $t \rightarrow \infty$
(i.e., the system is globally asymptotically stable)

intepretation:

- V is positive definite generalized energy function
- energy is always dissipated, except at 0

[from Boyd, ee363]

Proof:

Suppose trajectory $x(t)$ does not converge to zero.

$V(x(t))$ is decreasing and nonnegative, so it converges to, say, ϵ as $t \rightarrow \infty$.

Since $x(t)$ does not converge to 0, we must have $\epsilon > 0$, so for all t , $\epsilon \leq V(x(t)) \leq V(x(0))$.

$C = \{z | \epsilon \leq V(z) \leq V(x(0))\}$ is closed and bounded, hence compact. So \dot{V} (assumed continuous) attains its supremum on C , i.e., $\sup_{z \in C} \dot{V} = -a < 0$.

Since $\dot{V}(x(t)) \leq -a$ for all t , we have

$$V(x(T)) = V(x(0)) + \int_0^T \dot{V}(x(t)) dt \leq V(x(0)) - aT$$

which for $T > V(x(0))/a$ implies $V(x(T)) < 0$, a contradiction.

So every trajectory $x(t)$ converges to 0, i.e., $\dot{x} = f(x)$ is globally asymptotically stable.

[from Boyd, ee363]

Global invariant set Theorem. Assume that

- $V(x) \rightarrow \infty$ as $\|x\| \rightarrow \infty$.
- $\dot{V}(x) \leq 0$ over the whole state space.

Let R be the set of all points where $\dot{V}(x) = 0$, and let M be the largest invariant set in R . Then all solutions globally asymptotically converge to M as $t \rightarrow \infty$.

Example 1

$$\ddot{q} + \dot{q} + \sin q = u$$

$$u = \sin q + K_p(q^* - q) + K_d(0 - \dot{q})$$

Example 1 (solution)

$$\begin{aligned}\ddot{q} + \dot{q} + g(q) &= u \\ u &= g(q) + K_d(q^* - q) + K_d(0 - \dot{q})\end{aligned}$$

We choose $V = \frac{1}{2}K_p(q - q^*)^2 + \frac{1}{2}\dot{q}^2$.

This gives for \dot{V} :

$$\begin{aligned}\dot{V} &= K_p(q - q^*)\dot{q} + \dot{q}\dot{q} \\ &= K_p(q - q^*)\dot{q} + \dot{q}(K_p(q^* - q) - K_d\dot{q} - \dot{q}) \\ &= -(1 + K_d)\dot{q}\end{aligned}$$

Hence V satisfies: (i) $V(q) \geq 0$ and $= 0$ iff $q = q^*$, (ii) $\dot{V} \leq 0$. Since the arm cannot get "stuck" at any position such that $q \neq 0$ (which can be easily shown by noting that acceleration is non-zero in such situations), the robot arm must settle down at $\dot{q} = 0$ and $q = 0$, according to the invariant set theorem. Thus the system is globally asymptotically stable.

Example 2: PD controllers are stable for fully actuated manipulators

- (Slotine and Li, Example 3.21.)

A converse Lyapunov G.E.S. theorem

suppose there is $\beta > 0$ and M such that each trajectory of $\dot{x} = f(x)$ satisfies

$$\|x(t)\| \leq M e^{-\beta t} \|x(0)\| \text{ for all } t \geq 0$$

(called *global exponential stability*, and is stronger than G.A.S.)

then, there is a Lyapunov function that proves the system is exponentially stable, i.e., there is a function $V : \mathbf{R}^n \rightarrow \mathbf{R}$ and constant $\alpha > 0$ s.t.

- V is positive definite
- $\dot{V}(z) \leq -\alpha V(z)$ for all z

[from Boyd, ee363]

Proof of converse G.E.S. Lyapunov theorem

suppose the hypotheses hold, and define

$$V(z) = \int_0^{\infty} \|x(t)\|^2 dt$$

where $x(0) = z$, $\dot{x} = f(x)$

since $\|x(t)\| \leq M e^{-\beta t} \|z\|$, we have

$$V(z) = \int_0^{\infty} \|x(t)\|^2 dt \leq \int_0^{\infty} M^2 e^{-2\beta t} \|z\|^2 dt = \frac{M^2}{2\beta} \|z\|^2$$

(which shows integral is finite)

[from Boyd, ee363]

Finding Lyapunov functions

- there are many different types of Lyapunov theorems
- the key in all cases is to *find* a Lyapunov function and verify that it has the required properties
- there are several approaches to finding Lyapunov functions and verifying the properties

one common approach:

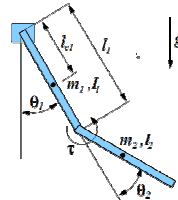
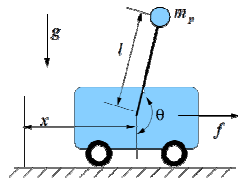
- decide form of Lyapunov function (e.g., quadratic), parametrized by some parameters (called a *Lyapunov function candidate*)
- try to find values of parameters so that the required hypotheses hold

[from Boyd, ee363]

Lyapunov recap

- Enables providing stability guarantees w/o solving the differential equations for all possible initial conditions!
- Tricky part: finding a Lyapunov function
- A lot more to it than we can cover in ½ lecture, but this should provide you with a starting point whenever you might need something like this in the future

Intermezzo on energy pumping



- An interesting and representative approach from non-linear control:
 - Energy pumping to swing up:
 - Write out energy of system $E(q, \dot{q})$
 - Write out time derivative of energy: $\dot{E}(q, \dot{q}, u)$
 - Choose u as a function of q and \dot{q} such that energy is steered towards required energy to reach the top
 - Local controller to stabilize at top --- we will see this aspect in detail later.

Energy pumping nicely described in Tedrake Chapter 3. Enjoy the optional read!

Forthcoming lectures

- Optimal control: provides general computational approach to tackle control problems---both under- and fully actuated.
 - Dynamic programming
 - Discretization
 - Dynamic programming for linear systems
 - Extensions to nonlinear settings:
 - Local linearization
 - Differential dynamic programming
 - Feedback linearization
 - Model predictive control (MPC)
 - Examples:

