Announcements

- Final project: 45% of the grade, 10% presentation, 35% write-up
  - Presentations: in lecture Dec 1 and 3
  - If you have constraints, inform me by email by Wednesday night, we will assign the others at random on Thursday
- PS2: due Friday 23:59pm.
- Tuesday 4-5pm 540 Cory: Hadas Kress-Gazit (Cornell)

High-level tasks to correct low-level robot control
In this talk I will present a formal approach to creating robot controllers that ensure the robot satisfies a given high level task. I will describe a framework in which a user specifies a complex and reactive task in Structured English. This task is then automatically translated, using temporal logic and tools from the formal methods world, into a hybrid controller. This controller is guaranteed to control the robot such that its motion and actions satisfy the intended task, in a variety of different environments.

CS 287: Advanced Robotics
Fall 2009

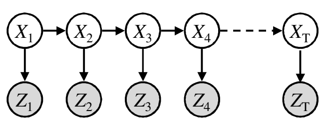
Lecture 23:
HMMs: Kalman filters, particle filters

Pieter Abbeel
UC Berkeley EECS

Hidden Markov Models

Joint distribution is assumed to be of the form:

$P(X_1 = x_1)\, P(Z_1 = z_1 \mid X_1 = x_1)\, P(X_2 = x_2 \mid X_1 = x_1)\, P(Z_2 = z_2 \mid X_2 = x_2) \ldots$
$P(X_T = x_T \mid X_{T-1} = x_{T-1})\, P(Z_T = z_T \mid X_T = x_T)$



Filtering in HMM

- Init $P(x_1)$ [e.g., uniformly]
- Observation update for time 0:
  $$P(x_1|z_1) \propto P(z_1|x_1)P(x_1)$$
- For t = 1, 2, …
  - Time update
    $$P(x_{t+1}|z_{1:t}) = \int_{x_t} P(x_{t+1}|x_t)P(x_t|z_{1:t})dx_t$$
  - Observation update
    $$P(x_{t+1}|z_{1:t+1}) \propto P(z_{t+1}|x_{t+1})P(x_{t+1}|z_{1:t})$$
- For discrete state / observation spaces: simply replace integral by summation

Discrete-time Kalman Filter

Estimates the state x of a discrete-time controlled process that is governed by the linear stochastic difference equation

$$x_t = A_t x_{t-1} + B_t u_t + \varepsilon_t$$

with a measurement

$$z_t = C_t x_t + \delta_t$$

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Kalman Filter Algorithm

Algorithm **Kalman_filter**(μ_{t-1}, Σ_{t-1}, u_t, z_t):

Prediction:

$$\bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$$
$$\bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$$

Correction:

$$K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$
$$\mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$$
$$\Sigma_t = (I - K_t C_t)\bar{\Sigma}_t$$

Return μ_t, Σ_t

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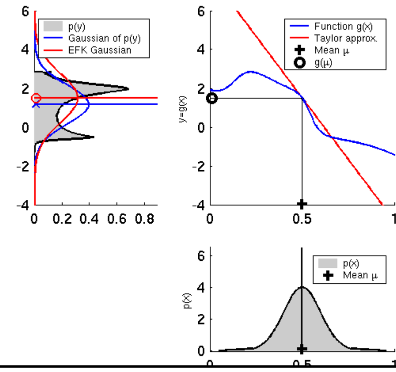
Nonlinear systems

$$x_t = g(u_t, x_{t-1}) + noise$$

$$z_t = h(x_t) + noise$$

Extended Kalman filter (EKF)



EKF Linearization: First Order Taylor Series Expansion

- Prediction:

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}} (x_{t-1} - \mu_{t-1})$$

$$g(u_t, x_{t-1}) \approx g(u_t, \mu_{t-1}) + G_t \, (x_{t-1} - \mu_{t-1})$$

- Correction:

$$h(x_t) \approx h(\bar{\mu}_t) + \frac{\partial h(\bar{\mu}_t)}{\partial x_t} (x_t - \bar{\mu}_t)$$

$$h(x_t) \approx h(\bar{\mu}_t) + H_t \, (x_t - \bar{\mu}_t)$$

EKF Algorithm

Extended_Kalman_filter($\mu_{t-1}, \Sigma_{t-1}, u_t, z_t$):

Prediction:

$$\bar{\mu}_t = g(u_t, \mu_{t-1}) \quad \longleftarrow \quad \bar{\mu}_t = A_t \mu_{t-1} + B_t u_t$$

$$\bar{\Sigma}_t = G_t \Sigma_{t-1} G_t^T + R_t \quad \longleftarrow \quad \bar{\Sigma}_t = A_t \Sigma_{t-1} A_t^T + R_t$$

Correction:

$$K_t = \bar{\Sigma}_t H_t^T (H_t \bar{\Sigma}_t H_t^T + Q_t)^{-1} \quad \longleftarrow \quad K_t = \bar{\Sigma}_t C_t^T (C_t \bar{\Sigma}_t C_t^T + Q_t)^{-1}$$

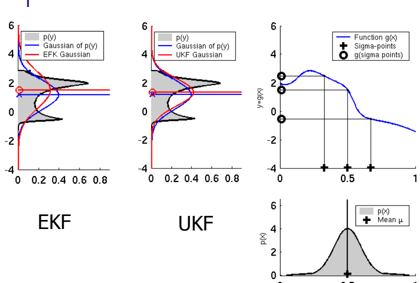
$$\mu_t = \bar{\mu}_t + K_t (z_t - h(\bar{\mu}_t)) \quad \longleftarrow \quad \mu_t = \bar{\mu}_t + K_t (z_t - C_t \bar{\mu}_t)$$

$$\Sigma_t = (I - K_t H_t) \bar{\Sigma}_t \quad \longleftarrow \quad \Sigma_t = (I - K_t C_t) \bar{\Sigma}_t$$

Return μ_t, Σ_t

$$H_t = \frac{\partial h(\bar{\mu}_t)}{\partial x_t} \qquad G_t = \frac{\partial g(u_t, \mu_{t-1})}{\partial x_{t-1}}$$

Linearization via Unscented Transform



EKF

UKF

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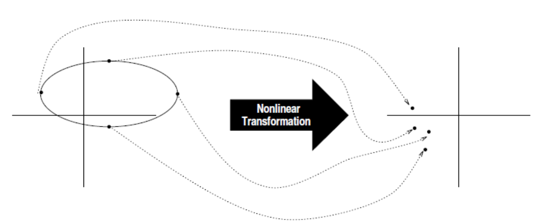
UKF Sigma-Point Estimate (4)



UKF intuition why it can perform better

- Assume we know the distribution over X and it has a mean \bar{x}
- Y = f(X)

$$\mathbf{f}[\mathbf{x}] = \mathbf{f}[\bar{\mathbf{x}} + \boldsymbol{\delta}\mathbf{x}]$$
$$= \mathbf{f}[\bar{\mathbf{x}}] + \nabla\mathbf{f}\boldsymbol{\delta}\mathbf{x} + \frac{1}{2}\nabla^2\mathbf{f}\boldsymbol{\delta}\mathbf{x}^2 + \frac{1}{3!}\nabla^3\mathbf{f}\boldsymbol{\delta}\mathbf{x}^3 + \frac{1}{4!}\nabla^4\mathbf{f}\boldsymbol{\delta}\mathbf{x}^4 + \cdots$$

$$\bar{\mathbf{y}} = \mathbf{f}[\bar{\mathbf{x}}] + \frac{1}{2}\nabla^2\mathbf{f}\,\mathbf{P}_{xx} + \frac{1}{2}\nabla^4\mathbf{f}\,\mathrm{E}\left[\boldsymbol{\delta}\mathbf{x}^4\right] + \cdots$$

$$\mathbf{P}_{yy} = \nabla\mathbf{f}\,\mathbf{P}_{xx}(\nabla\mathbf{f})^T + \frac{1}{2\times 4!}\nabla^2\mathbf{f}\left(\mathrm{E}\left[\boldsymbol{\delta}\mathbf{x}^4\right] - \mathrm{E}\left[\boldsymbol{\delta}\mathbf{x}^2\mathbf{P}_{yy}\right] - \mathrm{E}\left[\mathbf{P}_{yy}\boldsymbol{\delta}\mathbf{x}^2\right] + \mathbf{P}_{yy}^2\right)(\nabla^2\mathbf{f})^T +$$
$$\frac{1}{3!}\nabla^3\mathbf{f}\mathrm{E}\left[\boldsymbol{\delta}\mathbf{x}^4\right](\nabla\mathbf{f})^T + \cdots.$$

[Julier and Uhlmann, 1997]

UKF intuition why it can perform better

- Assume
  - 1. We represent our distribution over x by a set of sample points.
  - 2. We propagate the points directly through the function f.
- Then:
  - We don't have any errors in f !!
  - The accuracy we obtain can be related to how well the first, second, third, … moments of the samples correspond to the first, second, third, … moments of the true distribution over x.

Self-quiz

- When would the UKF significantly outperform the EKF?

Original unscented transform

- Picks a minimal set of sample points that match 1st, 2nd and 3rd moments of a Gaussian:

$$\begin{aligned}
\boldsymbol{\mathcal{X}}_0 &= \bar{\mathbf{x}} & W_0 &= \kappa/(n+\kappa) \\
\boldsymbol{\mathcal{X}}_i &= \bar{\mathbf{x}} + \left(\sqrt{(n+\kappa)\mathbf{P}_{xx}}\right)_i & W_i &= 1/2(n+\kappa) \\
\boldsymbol{\mathcal{X}}_{i+n} &= \bar{\mathbf{x}} - \left(\sqrt{(n+\kappa)\mathbf{P}_{xx}}\right)_i & W_{i+n} &= 1/2(n+\kappa)
\end{aligned}$$

- \bar{x} = mean, P_{xx} = covariance, i → i'th row, $x \in \Re^n$
- \kappa : extra degree of freedom to fine-tune the higher order moments of the approximation; when x is Gaussian, n+\kappa = 3 is a suggested heuristic

[Julier and Uhlmann, 1997]

Unscented Kalman filter

- Dynamics update:
  - Can simply use unscented transform and estimate the mean and variance at the next time from the sample points
- Observation update:
  - Use sigmapoints from unscented transform to compute the covariance matrix between x_t and z_t. Then can do the standard update.

Algorithm Unscented_Kalman_filter$(\mu_{t-1}, \Sigma_{t-1}, u_t, z_t)$**:**

1. $\mathcal{X}_{t-1} = (\mu_{t-1} \quad \mu_{t-1} + \gamma\sqrt{\Sigma_{t-1}} \quad \mu_{t-1} - \gamma\sqrt{\Sigma_{t-1}})$
2. $\bar{\mathcal{X}}_t^* = g(\mu_t, \mathcal{X}_{t-1})$
3. $\bar{\mu}_t = \sum_{i=0}^{2n} w_m^{[i]} \bar{\mathcal{X}}_t^{*[i]}$
4. $\bar{\Sigma}_t = \sum_{i=0}^{2n} w_c^{[i]} (\bar{\mathcal{X}}_t^{*[i]} - \bar{\mu}_t)(\bar{\mathcal{X}}_t^{*[i]} - \bar{\mu}_t)^\top + R_t$
5. $\bar{\mathcal{X}}_t = \left(\bar{\mu}_t \quad \bar{\mu}_t + \gamma\sqrt{\bar{\Sigma}_t} \quad \bar{\mu}_t - \gamma\sqrt{\bar{\Sigma}_t}\right)$
6. $\bar{\mathcal{Z}}_t = h(\bar{\mathcal{X}}_t)$
7. $\hat{z}_t = \sum_{i=0}^{2n} w_m^{[i]} \bar{\mathcal{Z}}_t^{[i]}$
8. $S_t = \sum_{i=0}^{2n} w_c^{[i]} \left(\bar{\mathcal{Z}}_t^{[i]} - \hat{z}_t\right)\left(\bar{\mathcal{Z}}_t^{[i]} - \hat{z}_t\right)^\top + Q_t$
9. $\bar{\Sigma}_t^{x,z} = \sum_{i=0}^{2n} w_c^{[i]} \left(\bar{\mathcal{X}}_t^{[i]} - \bar{\mu}_t\right)\left(\bar{\mathcal{Z}}_t^{[i]} - \hat{z}_t\right)^\top$
10. $K_t = \bar{\Sigma}_t^{x,z} S_t^{-1}$
11. $\mu_t = \bar{\mu}_t + K_t(z_t - \hat{z}_t)$
12. $\Sigma_t = \bar{\Sigma}_t - K_t S_t K_t^\top$
13. **return** μ_t, Σ_t

[Table 3.4 in Probabilistic Robotics]

UKF Summary

- Highly efficient: Same complexity as EKF, with a constant factor slower in typical practical applications
- Better linearization than EKF: Accurate in first two terms of Taylor expansion (EKF only first term) + capturing more aspects of the higher order terms
- Derivative-free: No Jacobians needed
- Still not optimal!

Particle filters motivation

- Particle filters are a way to **efficiently** represent **non-Gaussian distribution**

- Basic principle
  - Set of state hypotheses ("particles")
  - Survival-of-the-fittest

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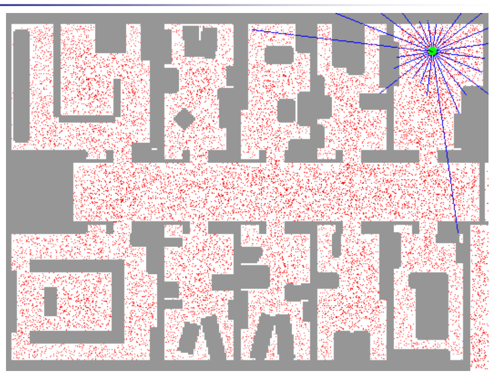
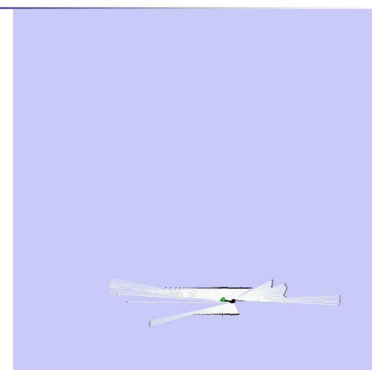
Sample-based Localization (sonar)

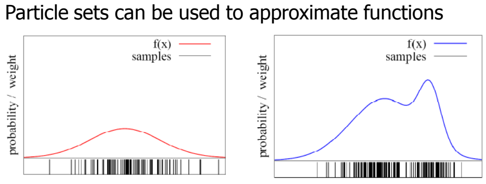


FastSLAM [particle filter + Rao-Blackwellization + occupancy grid mapping + scan matching based odometry]



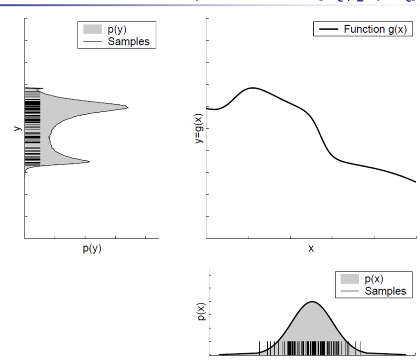
Sample-based Mathematical Description of Probability Distributions

- Particle sets can be used to approximate functions



- The more particles fall into an interval, the higher the probability of that interval
- If a continuous density needs to be extracted → can place a narrow Gaussian at the location of each particle
- How to efficiently draw samples in an HMM?

Dynamics update with sample representation of distribution: sample from $P(x_{t+1} \mid x_t)$



Observation update

- Goal: go from a sample-based representation of

  $P(x_{t+1} \mid z_1, \ldots, z_t)$

  to a sample-based representation of

  $P(x_{t+1} \mid z_1, \ldots, z_t, z_{t+1}) =$

  $C * P(x_{t+1} \mid z_1, \ldots, z_t) * P(z_{t+1} \mid x_{t+1})$

Importance sampling

- Interested in estimating:

$$
\begin{aligned}
\mathrm{E}_{X\sim P}[f(X)] &= \int_x f(x)P(x)dx \\
&= \int_x f(x)P(x)\frac{Q(x)}{Q(x)}dx \quad \text{if} Q(x)=0 \Rightarrow P(x)=0 \\
&= \int_x f(x)\frac{P(x)}{Q(x)}Q(x)dx \\
&= \mathrm{E}_{X\sim Q}[\frac{P(X)}{Q(X)}f(X)] \\
&\approx \frac{1}{m}\sum_{i=1}^{m}\frac{P(x^{(i)})}{Q(x^{(i)})}f(x^{(i)}) \quad \text{with } x^{(i)}\sim Q
\end{aligned}
$$

- ***Hence we could sample from an alternative distribution Q and simply re-weight the samples == Importance Sampling***

Sequential Importance Sampling (SIS) Particle Filter

- Sample $x^{(1)}_1, x^{(2)}_1, \ldots, x^{(N)}_1$ from $P(X_1)$
- Set $w^{(i)}_1 = 1$ for all $i=1,\ldots,N$
- For $t=1, 2, \ldots$
  - Dynamics update:
    - For $i=1, 2, \ldots, N$
      - Sample $x^{(i)}_t$ from $P(X_{t+1} \mid X_t = x^{(i)}_t)$
  - Observation update:
    - For $i=1, 2, \ldots, N$
      - $w^{(i)}_{t+1} = w^{(i)}_t * P(z_{t+1} \mid X_{t+1} = x^{(i)}_{t+1})$
- At any time t, the distribution is represented by the weighted set of samples

  $\{ <x^{(i)}_t, w^{(i)}_t> ; i=1,\ldots,N\}$

SIS particle filter major issue

- The resulting samples are only weighted by the evidence
- The samples themselves are never affected by the evidence

→ Fails to concentrate particles/computation in the high probability areas of the distribution $P(x_t \mid z_1, \ldots, z_t)$

Resampling

- At any time t, the distribution is represented by the weighted set of samples

  $\{ <x^{(i)}_t, w^{(i)}_t> ; i=1,\ldots,N\}$

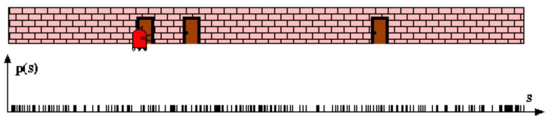
→ Sample N times from the set of particles

→ The probability of drawing each particle is given by its importance weight

→ More particles/computation focused on the parts of the state space with high probability mass

1. Algorithm **particle_filter**(S_{t-1}, u_{t-1} z_t):
2. $S_t = \emptyset, \quad \eta = 0$
3. **For** $i = 1 \ldots n$ ***Generate new samples***
4. Sample index $j(i)$ from the discrete distribution given by w_{t-1}
5. Sample x_t^i from $p(x_t \mid x_{t-1}, u_{t-1})$ using $x_{t-1}^{j(i)}$ and u_{t-1}
6. $w_t^i = p(z_t \mid x_t^i)$ ***Compute importance weight***
7. $\eta = \eta + w_t^i$ ***Update normalization factor***
8. $S_t = S_t \cup \{< x_t^i, w_t^i >\}$ ***Insert***
9. **For** $i = 1 \ldots n$
10. $w_t^i = w_t^i / \eta$ ***Normalize weights***

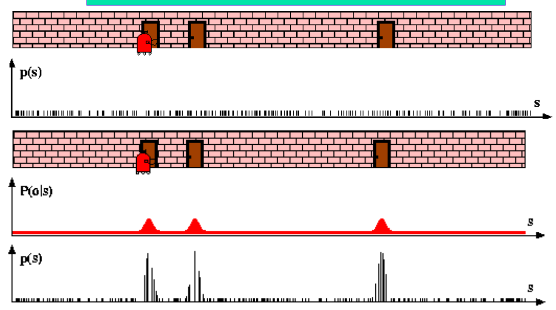
Particle Filters



Sensor Information: Importance Sampling

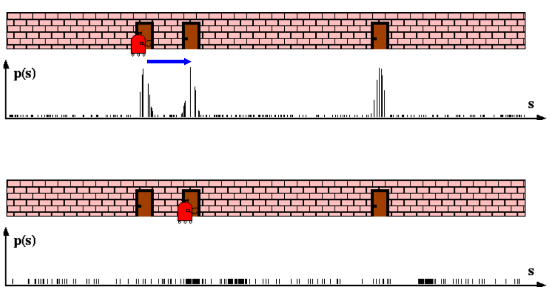
$$Bel(x) \leftarrow \alpha\, p(z \mid x)\, Bel^{-}(x)$$

$$w \leftarrow \frac{\alpha\, p(z \mid x)\, Bel^{-}(x)}{Bel^{-}(x)} = \alpha\, p(z \mid x)$$



Robot Motion

$$Bel^{-}(x) \leftarrow \int p(x \mid u.x')\, Bel(x')\, \mathrm{d}x'$$



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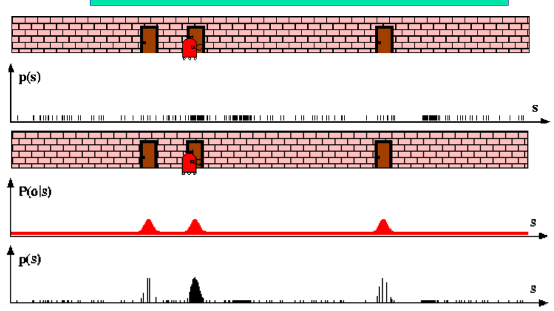

Robot Motion

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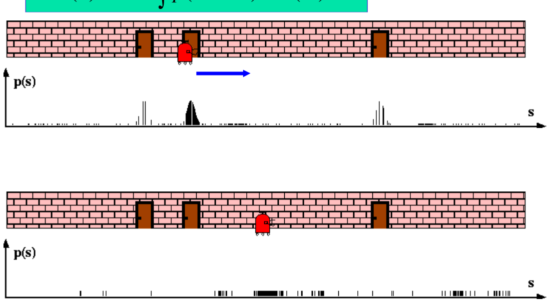


Resampling issue

- Loss of samples …

Low Variance Resampling

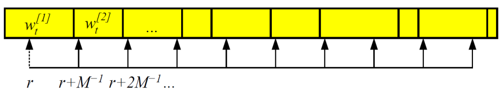


Figure 4.7 Principle of the low variance resampling procedure. We choose a random number r and then select those particles that correspond to $u = r + (m-1) \cdot M^{-1}$ where $m = 1, \ldots, M$.

Low Variance Resampling

- Advantages:
  - More systematic coverage of space of samples
  - If all samples have same importance weight, no samples are lost
  - Lower computational complexity

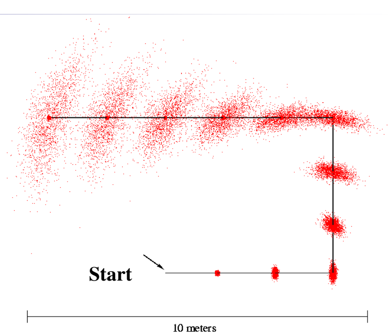
Regularization

- If no dynamics noise → all particles will start to coincide
- → regularization: resample from a (narrow) Gaussian around the particle

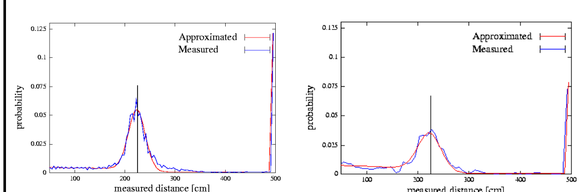
Mobile Robot Localization

- Each particle is a potential pose of the robot
- Proposal distribution is the motion model of the robot (prediction step)
- The observation model is used to compute the importance weight (correction step)

Motion Model Reminder



Proximity Sensor Model Reminder

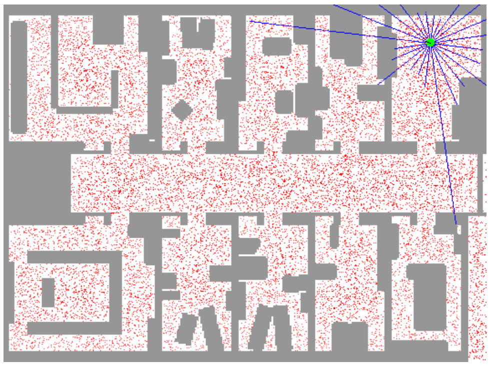


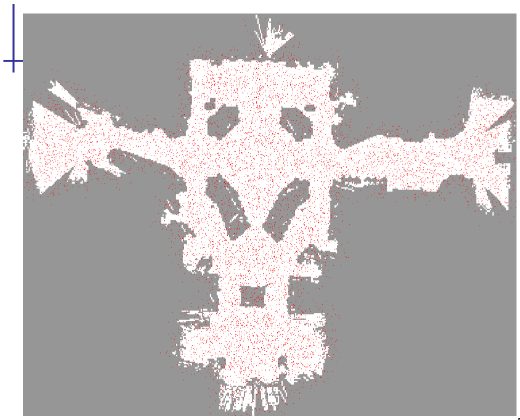
Laser sensor

Sonar sensor

Note: sensor model is not Gaussian at all!

Sample-based Localization (sonar)





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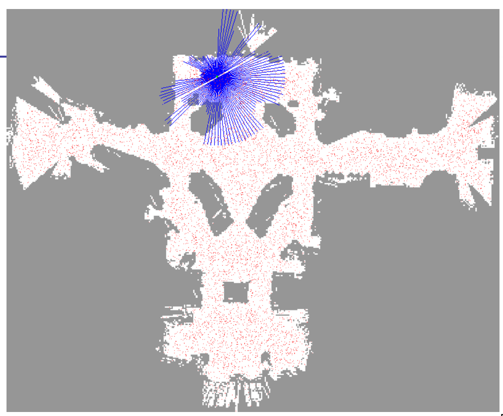

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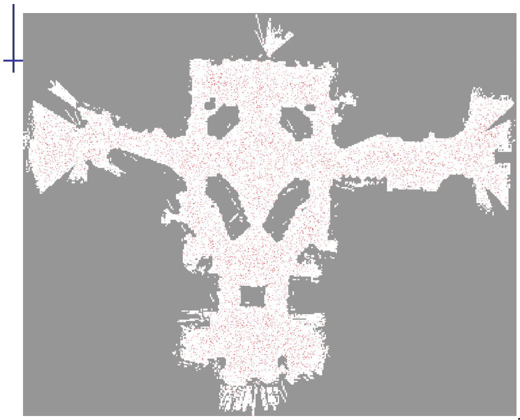

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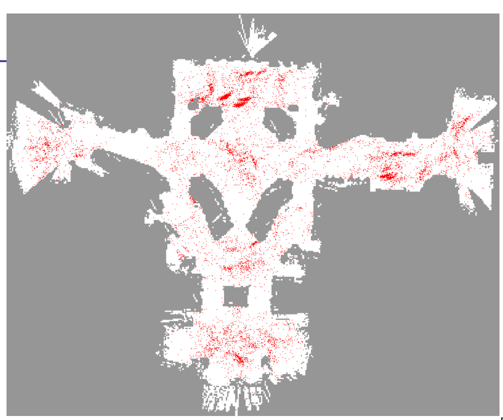

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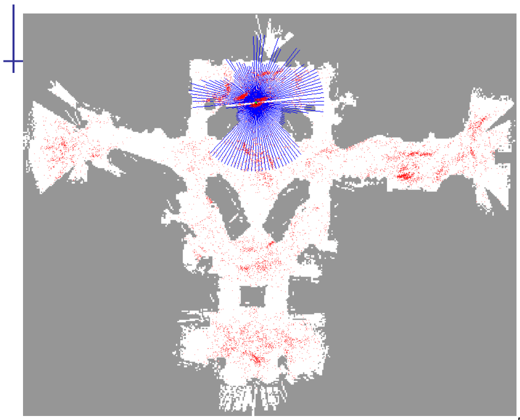

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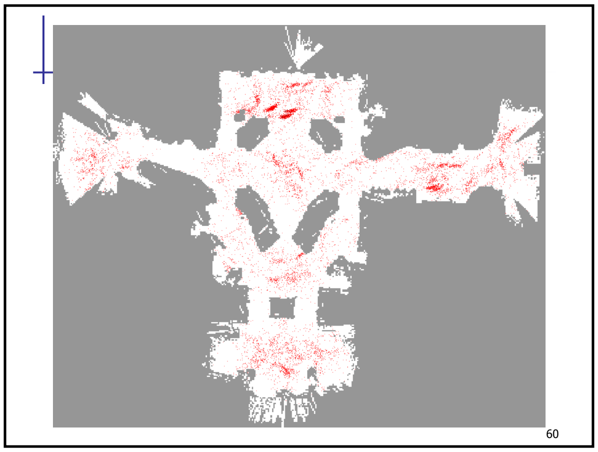
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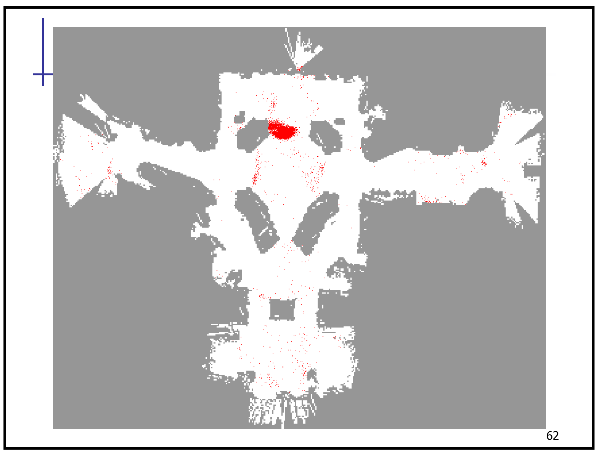
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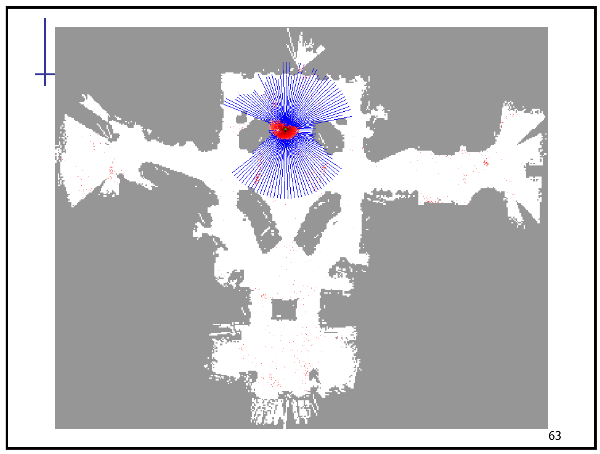
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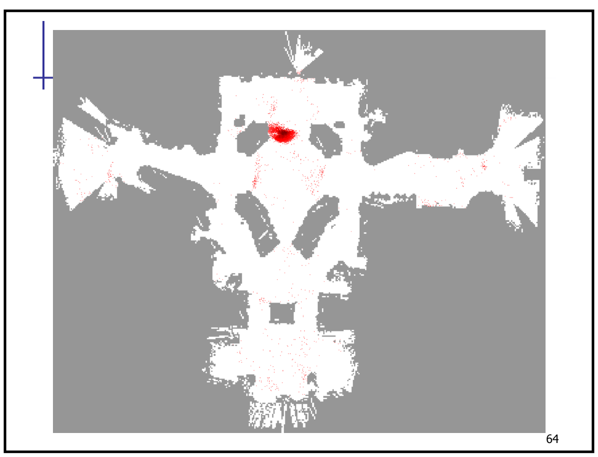
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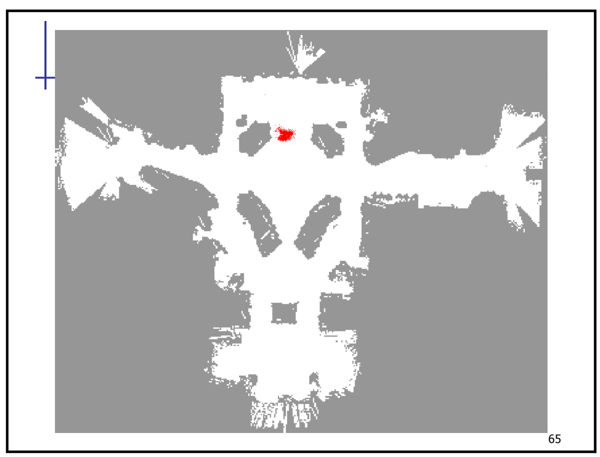
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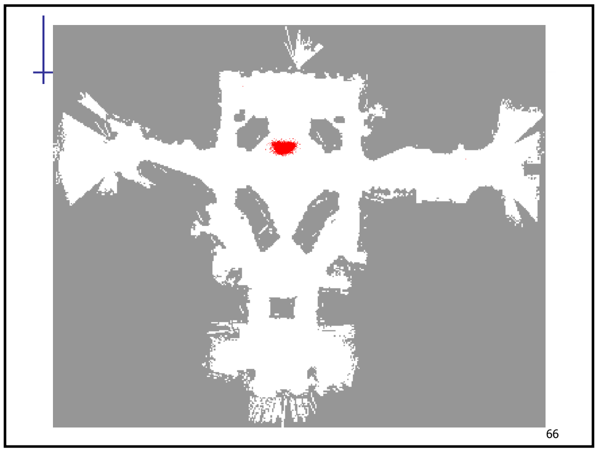
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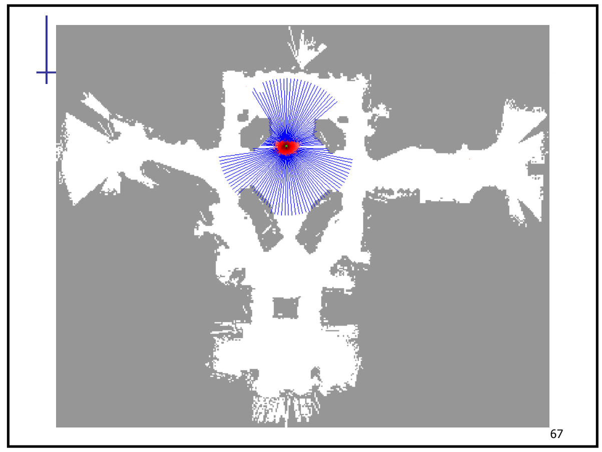
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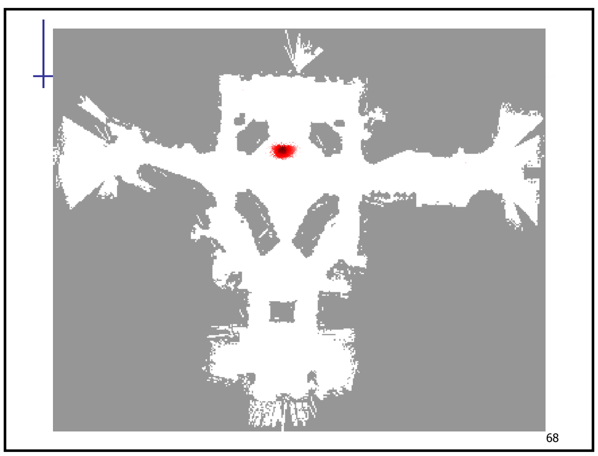
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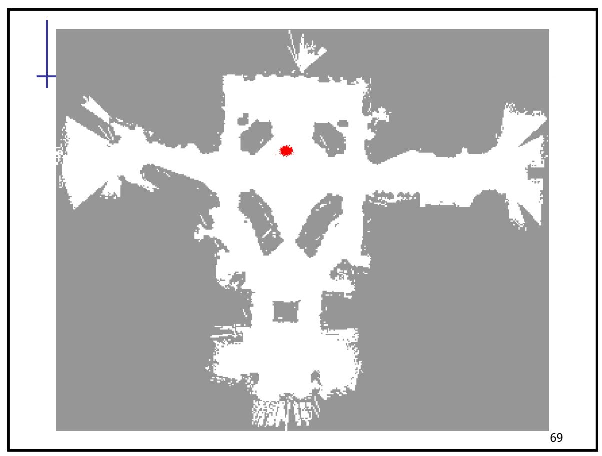
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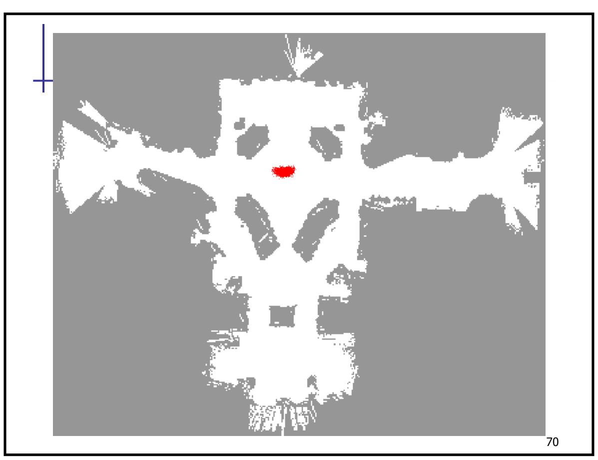
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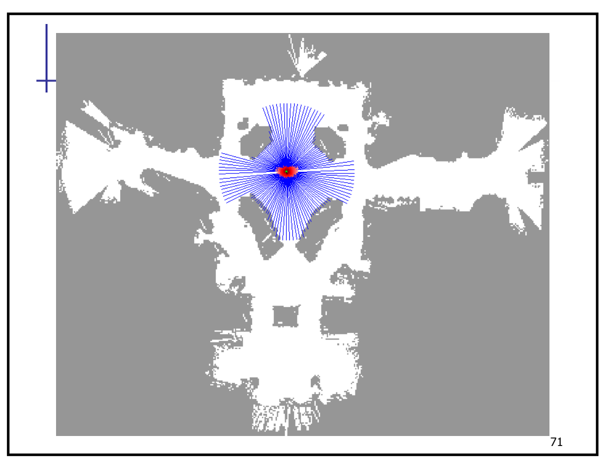
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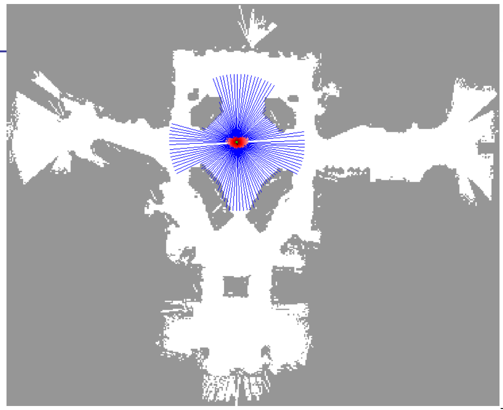

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Summary – Particle Filters

- Particle filters are an implementation of recursive Bayesian filtering
- They represent the posterior by a set of weighted samples
- They can model non-Gaussian distributions
- Proposal to draw new samples
- Weight to account for the differences between the proposal and the target
- Monte Carlo filter, Survival of the fittest, Condensation, Bootstrap filter

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Summary – PF Localization

- In the context of localization, the particles are propagated according to the motion model.
- They are then weighted according to the likelihood of the observations.
- In a re-sampling step, new particles are drawn with a probability proportional to the likelihood of the observation.

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