An Explicit Construction of an Expander Family (Margulis-Gaber-Galil) [M75, GG81]

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`The purpose of these notes is to help the students (including, most importantly, me) during the presentation, so informalities and inaccuracies are to be expected.`

1 Introduction

We present an simple construction of an expander family due to [M75] and [GG81]. As is often the case, a simple construction requires complicated analysis, but we present the analysis of [T14, 6.2], which is composed of three straightforward steps and is beautiful in its own right. As you will see, it also incorporates and generalizes many of the concepts we saw in the course.

1.1 Overview

Our alleged expander will be $\{G_n\}_{n\in\mathbb{N}}$, where G_n is a graph on vertices $\mathbb{Z}_n \times \mathbb{Z}_n$, where (a, b) is connected to

$$S(a,b), S^{-1}(a,b), T(a,b), T^{-1}(a,b), (a\pm 1, b), (a, b\pm 1)$$

where $S(a,b) \coloneqq (a, a+b)$, $T(a,b) \coloneqq (a+b, b)$, $S^{-1}(a,b) = (a, b-a)$, $T^{-1}(a,b) = (a-b, b)$ with addition and subtraction modulus n.

To show that $\{G_n\}_{n\in\mathbb{N}}$ is an expander family, we must show that

$$\lambda_2(G_n) = \min_{f\perp\mathbf{1}, f\neq 0} \frac{\varepsilon(f)}{\mathrm{Var}(f)} = \min_{f\perp\mathbf{1}, f\neq 0} \frac{\mathbb{E}_{u\sim v}\left[(f(u)-f(v))^2\right]}{\mathbb{E}_{u\leftarrow\pi}\left[f(u)^2\right]} = \Omega(1)$$

and we do this by constructing an infinite version of this graph which would be easier to analyze in terms of expansion, and showing that the infinite version has similar expansion properties to G_n. In fact, we will have an additional (infinite) intermediate graph.

For a more detailed explanation, we must introduce two additional actors.

- The graph family $\{R_n\}_{n\in\mathbb{N}}$. Each R_n is a graph on vertices $[0,n)\times[0,n)$, where each (x, y) is connected to $S(x,y), S^{-1}(x,y), T(x,y), T^{-1}(x,y)$.
- Graph Z on vertices $\mathbb{Z}\times\mathbb{Z}\backslash\{0,0\}$, Each vertex (a, b) is connected to $S(x,y), S^{-1}(x,y), T(x,y), T^{-1}(x,y)$, only now we define S and T using regular addition (not addition modulus n).

Informally[1], we will then prove each of the following inequalities,

$$\Theta(\lambda_2(G_n)) \geq \lambda_2(R_n) \geq \lambda_2(Z) = \Omega(1)$$

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[1] As we have not yet defined $\lambda_2(R_n)$ and $\lambda_2(Z)$.

2 The countable case

Let's start at the end. The notion of conductence of a graph can be generalized to countably infinite graphs by letting

$$\Phi(G) := \inf_{A \text{ finite and nonempty}} \frac{\left|E\left(A, \overline{A}\right)\right|}{|A|}$$

We then reason about Z's conductence.

Proposition 2.1. $\Phi(Z) \geq 1/3$.

Proof. Let A be a finite nonempty subset of $\mathbb{Z}^2 \setminus \{0,0\}$. Let A_i be the intersection of A with the i-th quadrant of $\mathbb{Z}^2$, and let A_0 be the intersection of A with each axis.

Claim 2.2. $\left|E\left(A \setminus A_0, \overline{A}\right)\right| \geq |A \setminus A_0| = |A| - |A_0|$

Proof. Consider A_1. The following three properties hold:

- $|S(A_1)| = |T(A_1)| = |A_1|$
  - $S(A_1) \cap T(A_1) = \emptyset$. Otherwise we'd have $(a, a+b) = (a'+b', b')$ which implies that $b = -a'$, which the fact that (a,b) and (a',b') reside in the first quadrant.
  - $S(A_1)$ and $T(A_1)$ are contained in the first quadrant.

$S(A_1) \sqcup T(A_1)$ is twice A_1's size and is contained in the first quadrant, so at least half of its elemnts aren't in A. Since the set of edges with one point in A_1 contains $S(A_1) \sqcup T(A_1)$, this means that there are at least $|A_1|$ edges leaving A_1 and landing in $\overline{A}$.

Repeating the above claim for A_2, A_3 and A_4 (while considering $\left(S^{-1}, T^{-1}\right)$, (S,T) and $\left(S^{-1}, T^{-1}\right)$ respectively) gives that for all $i \in [4]$, $|A_i|$ of edges leaving A_i land in $\overline{A}$. In other words,

$$\left|E\left(A \setminus A_0, \overline{A}\right)\right| = \left|\bigsqcup_i E\left(A_i, \overline{A}\right)\right| = \sum_i \left|E\left(A_i, \overline{A}\right)\right| \geq \sum_i |A_i| = |A \setminus A_0|$$

□

Claim 2.3. $\left|E\left(A_0, \overline{A}\right)\right| \geq 5|A_0| - 3|A|$

Proof. Each vertex in A_0 has two outgoing edges that land outside A_0 (e.g. $(a,0)$ is connected to $(a, \pm a)$).[2] Some of these edges land in $A \setminus A_0$ and some land in $\overline{A}$. $A \setminus A_0$ can account for at most $4|A \setminus A_0|$ edges, therefore the number of edges landing in $\overline{A}$ is at least $2|A_0| - 4|A \setminus A_0|$. But using the previous claim we see that $A \setminus A_0$ can account for at most $3|A \setminus A_0|$ edges, as at least $|A \setminus A_0|$ edges come from $\overline{A}$ (and specifically do not come from A_0), giving us that $\left|E\left(A_0, \overline{A}\right)\right| \geq 2|A_0| - 3|A \setminus A_0| = 5|A_0| - 3|A|$. □

To conclude the proof, add 5/6 of the first inequality to 1/6 of the second. Formally, observe that

$$\left|E\left(A, \overline{A}\right)\right| = \left|E\left(A \setminus A_0, \overline{A}\right)\right| + \left|E\left(A_0, \overline{A}\right)\right| \geq \frac{5}{6}\left|E\left(A \setminus A_0, \overline{A}\right)\right| + \frac{1}{6}\left|E\left(A_0, \overline{A}\right)\right| \geq \frac{5|A| - 5|A_0| + 5|A_0| - 3|A|}{6} = \frac{1}{3}|A|$$

□

Similarly to the finite case (following the Courant-Fischer theorem), for a countably infinite graph G we define $\lambda_2(G)$ to be the infimum of the Rayleigh quotient taken over all zero-meaned functions (that are nice enough)[3].

$$\lambda_2(G) := \inf \frac{\sum_{(u,v) \in E(G)} (f(u) - f(v))^2}{\sum_{u \in V(G)} f(u)^2}$$

[2] The other two edges are self-loops.

[3] We could have defined a Laplacian for countably infinite graphs and proven that its second eigenvalue is equal to said infimum, but this is not needed for our proof.

Where the infimum is taken over $f : \mathbb{Z}^2 \to \mathbb{R}$ such that $\sum_{c\in\mathbb{Z}^2} f(c)^2 < \infty, \sum_{c\in\mathbb{Z}^2} f(c) = 0, f \not\equiv 0$. The dilligent reader can try to generalize the total variation and variance over finite graphs to countably infinite ones, and find that their ratio (i.e. the Rayleigh quotient) is exactly as above. Plugging Z into our definition, we get

$$\lambda_2(Z) := \inf \cancel{\frac{1}{2}} \cdot \frac{\sum_{u\in\mathbb{Z}^2} \left[(f(u) - f(S(u)))^2 + (f(u) - f(T(u)))^2\right]}{\sum_{u\in\mathbb{Z}^2} f(u)}$$

We ignore the factor of 1/2 (and keep it in mind for the final tally if we wish to make it). Following these definition, it is natural to prove a Cheeger inequality for them, and indeed such a relation holds.

Fact 2.4. *For any countably infinite d-regular graph G,*

$$\Phi(G) \leq \sqrt{2d\lambda_2(G)}$$

The proof is similiar to the proof for the finite case we saw in class (use Fiedler's algorithm). From this and the above we get that $\lambda_2(G) \geq \Phi(G)^2/16 \geq 1/144$.

3 Countable to continuous

This is where things get interesting. Again, we define an analogue to the Rayleigh quotient, this time for continuous (uncountable) graphs. To avoid many technicalities, not only do we skip the definition of a Laplacian operator (and proof of the eigenvalue and Rayleigh qutient correspondence), but also plug R_n straight into the definition from the get-go.[4] Let L_2 denote . Let

$$\lambda_2(R_n) := \inf_{f\in L_2} \frac{\int_{[0,n)^2} (f(u) - f(S(u)))^2 + (f(u) - f(T(u)))^2 \,\mathrm{d}u}{\int_{[0,n)^2} f(u)^2 \,\mathrm{d}u}$$

Where the infimum is taken over the space of functions $f : [0,n)^2 \to \mathbb{R}$ such that $\int_{[0,n)^2} f(z)^2 \,\mathrm{d}z$ is well-defined (i.e. f is Lebesgue integrable), finite and nonzero, and that $\int_{[0,n)} f(z) \,\mathrm{d}z = 0$.

Proposition 3.1.

$$\lambda_2(R_n) \geq \lambda_2(Z)$$

Proof. Each $f \in L_2$ has a Fourier decomposition given by $f(z) = \sum_{c\in\mathbb{Z}^2} \widehat{f}(c) \cdot \chi_c(z)$, where

$$\chi_{a,b}(x,y) = \frac{1}{n} e^{2\pi i(ax+by)} \qquad \widehat{f}(c) = \langle f, \chi_c \rangle = \int_{[0,n)^2} f(z)\, \chi_c(z) \,\mathrm{d}z$$

Our goal is to show a correspondance between the expression minimized in the LHS and the expression minimized in the RHS. Specifically, we will show that the Rayleigh quotient of $f : [0,n)^2 \to \mathbb{R}$ (w.r.t R_n) is equal to the Rayleigh quotient of $\widehat{f} : \mathbb{Z}^2 \to \mathbb{R}$ (w.r.t Z), which gives the requires result.

For the denomnators, we use the Parseval equality (for this continuous product space) to get

$$\sum_{c\in\mathbb{Z}^2\setminus\{(0,0)\}} \widehat{f}(c)^2 = \sum_{c\in\mathbb{Z}^2} \widehat{f}(c)^2 = \int_{[0,n)^2} f(z)^2 \,\mathrm{d}z$$

where the leftmost equality is from the assumption that $\widehat{f}(0,0) = \int_{[0,n)^2} f(z) \,\mathrm{d}z = 0$.

For the numerators, let $s(z) := f(z) - f(S(z))$ and similarly $t(z) := f(z) - f(T(Z))$, then using Parseval's equality and linearity of Fourier coefficients (i.e. linearity of an inner product),

$$\int_{[0,n)^2} s(z)^2 + t(z)^2 \,\mathrm{d}z = \sum_{c\in\mathbb{Z}^2} \widehat{s}(c)^2 + \widehat{t}(c)^2 = \sum_{c\in\mathbb{Z}^2\setminus\{(0,0)\}} \left(\widehat{f}(c) - \widehat{f\circ S}(c)\right)^2 + \left(\widehat{f}(c) - \widehat{f\circ T}(c)\right)^2$$

[4] What's important is that we're be able to reason about this expression both now and in the following section.

To conclude, we observe that

$$\begin{aligned}\widehat{f\circ S}(a,b) &= \frac{1}{n}\int_{[0,n)^2} f(S(a,b))\,e^{2\pi i(ax+by)}\mathrm{d}(x,y) = \frac{1}{n}\int_{[0,n)^2} f(x,x+y)\,e^{2\pi i(ax+by)}\mathrm{d}(x,y) \\ \underset{y'=x+y}{=} &\ \frac{1}{n}\int_{[0,n)^2} f(x,y')\,e^{2\pi i\left((a-b)x+by'\right)} = \widehat{f}(a-b,b) = \widehat{f}\circ T^{-1}(a,b)\end{aligned}$$

and similarly $\widehat{f\circ T}(a,b) = \widehat{f}\circ S^{-1}(a,b)$.

As such,

$$\begin{aligned}\int_{[0,n)^2} s(z)^2 + t(z)^2\,\mathrm{d}z &= \sum_{c\in\mathbb{Z}^2}\left(\widehat{f}(c) - \widehat{f\circ S}(c)\right)^2 + \left(\widehat{f}(c) - \widehat{f\circ T}(c)\right)^2 \\ &= \sum_{c\in\mathbb{Z}^2}\left(\widehat{f}(c) - \widehat{f}\circ T^{-1}(c)\right)^2 + \left(\widehat{f}(c) - \widehat{f}\circ S^{-1}(c)\right)^2 \\ &= \sum_{c\in\mathbb{Z}^2}\left(\widehat{f}\circ T(c) - \widehat{f}(c)\right)^2 + \left(\widehat{f}\circ S(c) - \widehat{f}(c)\right)^2\end{aligned}$$

so the numerators are equal as well. □

4 $\lambda_2(G_n) \geq \Omega(\lambda_2(R_n))$

We have all the tools we need, so we can jump straight to the proof.

Proof. Let f be the minimizer of G_n's Rayleigh quotient. That is, $f:\mathbb{Z}_n^2\to\mathbb{R}$ such that $f\not\equiv 0$, $\sum_{u\in\mathbb{Z}_n^2} f(u) = 0$, and[5]

$$\lambda_2(G) = \frac{1}{4}\cdot\frac{\sum_{u\in\mathbb{Z}_n^2}(f(u)-f(S(u))^2 + (f(u)-f(T(u)))^2 + (f(u)-f(u+e_1))^2 + (f(u)-f(u+e_2))^2}{\sum_{u\in\mathbb{Z}_n^2} f(u)^2}$$

Again, we wish to show a correspondence between functions $f:\mathbb{Z}_n^2\to\mathbb{R}$ and functions $\widetilde{f}:[0,n)^2\to\mathbb{R}$. This time, we extend $f:\mathbb{Z}_n^2\to\mathbb{R}$ (nonzero, zero meaned) to all of $[0,n)^2$ by breaking $[0,n)^2$ into a grid of 1×1 squares, and letting the value of the $\widetilde{f}$ on each square be equal to the value of the f on the square's bottom-left vertex. That is, we let $\widetilde{f}(x,y) \coloneqq f(\lfloor x\rfloor, \lfloor y\rfloor)$.

Since the area of each square in the grid is exactly 1, then $\int_{[0,n)^2}\tilde{f}(z)\,\mathrm{d}z = \sum_{u\in\mathbb{Z}_n^2} f(u)$, and so the denominators are equal (up to factor 4).

We show that the numerators are equal. Let's focus on the left addend of the RHS, $\int_{[0,n)^2}(f(z)-f(S(z)))^2$. Integrating over each square (of the grid) seperately, we have

$$\int_{[0,n)^2}\left(\widetilde{f}(z) - \widetilde{f}(S(z))\right)^2\mathrm{d}z = \sum_{(a,b)\in\mathbb{Z}_n^2}\int_{[a,a+1)\times[b,b+1)}\left(\widetilde{f}(z) - \widetilde{f}(S(z))\right)^2\mathrm{d}z$$

Now, in a given square, the top-left to bottom-right diagonal splits the square in half, and in the bottom triangle it holds that $\lfloor a+b\rfloor = \lfloor a\rfloor + \lfloor b\rfloor$, whereas in the top triangle it holds that $\lfloor a+b\rfloor = \lfloor a\rfloor + \lfloor b\rfloor + 1$.

[5] In very few details, this follows from

$$\mathbb{E}_{u\sim v}\left[(f(u)-f(v))^2\right] = \frac{1}{8}\sum_{\phi\in\left\{S^{\pm1},T^{\pm1},(\cdot)\pm e_1,(\cdot)\pm e_2\right\}}\mathbb{E}_{u\leftarrow\pi}(f(u)-f(\phi(u)))^2 = \frac{1}{8}\sum_{\phi\in\{S,T,(\cdot)+e_1,(\cdot)+e_2\}} 2\mathbb{E}_{u\leftarrow\pi}\left[(f(u)-f(\phi(u)))^2\right]$$

and since π (G_n's stationary distribution) is uniform as G_n is d-regular.

As such,

$$\begin{aligned}
\int_{[a,a+1)\times[b,b+1)} \left(\widetilde{f}(z) - \widetilde{f}(S(z))\right)^2 \mathrm{d}z &= \frac{1}{2}\left((f(a,b) - f(a,a+b))^2 + (f(a,b) - f(a,a+b+1))^2\right) \\
\overset{c=(a,b)}{} &= \frac{1}{2}\left((f(c) - f(S(c)))^2 + (f(c) - f(S(c)+e_2))^2\right) \\
\overset{(\alpha-\gamma)^2 \le 2\left((\alpha-\beta)^2+(\beta-\gamma)^2\right)}{} &\le \frac{1}{2}\left(\begin{array}{c} (f(c) - f(S(c)))^2 + 2(f(c) - f((c)+e_2))^2 \\ +2(f(c+e_2) - f(S(c)+e_2))^2 \end{array}\right) \\
&= \frac{1}{2}(f(c) - f(S(c)))^2 + (f(c) - f((c)+e_2))^2 + (f(c+e_2) - f(S(c)+e_2))^2
\end{aligned}$$

now summing over all squares,

$$\begin{aligned}
\sum_{(a,b)\in\mathbb{Z}_n^2} \int_{[a,a+1)\times[b,b+1)} \left(\widetilde{f}(z) - \widetilde{f}(S(z))\right)^2 \mathrm{d}z &= \sum_{(a,b)\in\mathbb{Z}_n^2} \left(\begin{array}{c} \frac{1}{2}(f(c) - f(S(c)))^2 + (f(c) - f((c)+e_2))^2 \\ +(f(c+e_2) - f(S(c)+e_2))^2 \end{array}\right) \\
\text{Changing order of summation} &= \sum_{(a,b)\in\mathbb{Z}_n^2} \frac{3}{2}(f(c) - f(S(c)))^2 + (f(c) - f((c)+e_2))^2 \\
&\le \frac{3}{2} \sum_{(a,b)\in\mathbb{Z}_n^2} (f(c) - f(S(c)))^2 + (f(c) - f((c)+e_2))^2
\end{aligned}$$

Making the analgous observations for T, we have

$$\int_{[0,n)^2} \left(\widetilde{f}(z) - \widetilde{f}(T(z))\right)^2 \mathrm{d}z \le \frac{3}{2} \sum_{(a,b)\in\mathbb{Z}_n^2} (f(c) - f(T(c)))^2 + (f(c) - f((c)+e_1))^2$$

Summing both inequalities, we get that the numerator of G_n's Rayleigh quotient is at least two-thirds the numerator of R_n's Rayleigh quotient. Putting everything together, we have that

$$\lambda_2(G_n) \ge \frac{2}{3} \cdot \frac{1}{4} \lambda_2(R_n) = \frac{1}{6} \lambda_2(R_n)$$

□

References

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