FAT Algorithms seminar
Liquid Democracy [KMP18, GKMP18]

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`The purpose of these notes is to help the students (including, most importantly, me) during the presentation, so informalities and inaccuracies are to be expected.`

1 Prologue

The FAT terminology is used in terms of algorithm-based decision-making. The decisions being made are usually very specific ones, e.g "Show ad" or "Approve loan", but as a mental exercise we will measure FATness for general decision making, i.e deciding the policy of a society -- governance.

- Is democracy *fair*? The straightforward answer is "yes", as one of its pillars is equality alongside minority rights.
- Is democracy *accountable*? Here the answer is much less clear, and we are immediately tempted to answer negatively. Yes, there are mechanisms to enforce accountability on our elected leaders, but reality shows us time and again that they are unsatisfactory.
- Is democracy *transparent*? See previous.

One of the main things standing between representative democracy and accountability and transparency is the fact that upon election, leaders do not need to answer to the public for their entire term (other than the minor checks and balances, which are hardly enough). On the other hand, a direct democracy is not viable as we cannot hope to keep all citizens to be experts on all daily issues.

In a Liquid Democracy[1], citizens choose whether to delegate their vote or vote directly on each individual issue. Indeed this alternative seems more accountable and transparent than RepDem as it is easier to analyze individual decisions rather than entire terms of representatives. But as a decision making-mechanism, is it indeed more transparent? Surely it is more complex than RepDem, e.g since the transitive delegations must be analyzed and understood. This leads us to consider and analyze the different implementations of liquid democracy.

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[1]First examined by [Miller69]. James C. Miller was a senior official (budget director) under President Nixon.

Part I

Liquid Democracy: An Algorithmic Perspective[KMP18]

2 The model

2.1 The setting

We model a vote on a binary issue assuming grount truth (i.e one outcome is correct and the other is wrong). The model has the following components.

- A directed *social network* (V, E) where $(i, j) \in E$ indicates that i knows (of) j.
- A *competency* measure $p : V \to [0, 1]$, where $p(i)$ is the probability that voter i votes correctly on the issue at hand.
- An *approval parameter* $\alpha \in [0, 1)$. We say that i *approves* j if $p(j) > p(i) + \alpha$ or if $j = i$,[2] and let $A_G(i) := \{j | i \text{ approves } j\}$. We assume that voters delegate only to members of $A_G(i)$, where self-delegation is interpreted as direct voting. This assumption strengthens our upcoming negative result, and saves us from dealing with delegation cycles (though research in this are exists, see e.g [BBCV11])

2.2 Delegation Mechanisms

The above completely describes a setting for an election, and now we need to describe how the election is performed. A liquid democracy is implemented through a *delegation mechanism* M which takes as input $G = (V, E, p)$ and outputs a distribution on $A_G(i)$ for each i, which indicates the probability that i delegates their vote to each member of $A_G(i)$. The probability that M makes a correct decision on G is denoted by $P_M(G)$ and is described by the following random process:

1. Apply M to G to receive a distribution X_i on $A_G(i)$, for each i.
2. For each $i \in V$, sample j from X_i and let i delegate to j. This results in a(n acylclic) delegation subgraph. Each sink i of this subgraph has weight equal to the number of directed paths ending at i, including the 0-length path (i).
3. Each sink i votes correctly with probability $p(i)$, and incorrectly with probability $1 - p(i)$.
4. The election outcome is determined according to the weighted majority vote (with arbitrary tie breaking).

A mechanism class of significance is the class of *local* mechanisms. A mechanism M is *local* if for each i, the distribution M constructs on $A_G(i)$ is based only on $\Gamma_G(i)$ and $A_G(i)$.[3] The following are examples of local delegation mechanisms:

- *Direct*: Voters never delegate. That is, for each i the distribution X_i is i w.p 1. This mechanism is the implementation of direct democracy in our framework, and will be known as the *direct* mechanism and denoted by D.
- *Random*: Voters delegate to a uniformly random approved voter. That is, $X_i \sim U(A_G(i))$ for all i.
- *Greedy*: Voters delegate to an approved voter of highest competency. That is, $X_i \equiv \text{argmax}_{j \in A_G(i)} \{p(j)\}$ with arbitrary[4] or random tie breaking.

[2] In the original paper we do not define voters to always approve themselves, so don't be confused.

[3] Formally, M is local if there exists a function m that takes as input a star (endowed with competencies) and outputs a distribution on that star, and M is the application of m to the substar centered at i, for each i.

[4] Formally, we assume that $V = [n]$ and assume that m gets as input a *labled* star.

In a way, local mechanisms are the most natural and acceptable, as they correspond to an election in which each voter votes based only on their perspective (and assuming all voters follow the same decision-making process). In fact, non-local mechanisms are somewhat absurd, as they allow voters to choose their delegations based, perhaps, on the choices of other participants which they might not even know!

2.3 Criteria

We're interested in comparing how certain liquid democratic mechanisms fare against direct democracy. Denote by D the direct mechanism, that outputs a distribution that is i w.p 1, for each i. Let

$$\text{gain}(M, G) \coloneqq P_M(G) - P_D(G)$$

A mechanism can be "better" in one of two ways (and preferably both):

- A mechanism M satisfies *do no harm (DNH)* if for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ and all graphs G_n on n vertices, $\text{gain}(M, G) \geq -\varepsilon$.
- A mechanism M satisfies *positive gain (PG)* if there exists $\gamma > 0$ and $N \in \mathbb{N}$ such that for all $n \geq N$ there exists a graph G_n on n vertices with $\text{gain}(M, G) \geq \gamma$.

Comparing the two,

$$\begin{array}{cccccc} \text{DNH} & \boldsymbol{\forall \varepsilon > 0} & \exists N & \forall n \geq N & \boldsymbol{\forall G_n} & \text{gain}(M,G) \geq -\varepsilon \\ \text{PG} & \boldsymbol{\exists \gamma > 0} & \exists N & \forall n \geq N & \boldsymbol{\exists G_n} & \text{gain}(M,G) \geq \gamma \end{array}$$

So DNH requires that M is *no worse* that D on *all* instances (eventually), whereas PG asserts the *existance* of instances on which M *beats* D.

3 Impossibility for local mechanisms[KMP18]

Under the assumption that voters only delegate to more-competent neighbours, it seems obvious that liquid democracy is superior to direct one, in both senses described above. Intuitively, this should be true for local mechanisms as well, but we will soon see that this is not the case (and can never be). Let's start with a warm-up.

Example 3.1. Let $n > 1$ be odd, and let G_n be a star on n-vertices, that is $V = [n]$ and $E = \{(i, n)\}_{n \in [n]}$. Let $p(n) = 4/5$ and $p(i) = 2/3$ for $i \neq n$, and choose $\alpha = 0.1$.

First notice that $P_D(G_n) \xrightarrow[n\to\infty]{} 1$, in what is generally known as Condorect's Jury Theorem. Let Y_i be a random variable indicating that i voted correctly. Since $\mathbb{E}[Y_1] = 2/3$ and the Y_i are i.i.d for $i < n$, the weak law of large numbers[5] gives us

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{Y_1 + \cdots + Y_{n-1}}{n-1} \leq \frac{2}{3} - 0.01\right] = 0$$

So we have

$$\begin{aligned} \lim_{n\to\infty} P_D(G_n) &\geq \lim_{n\to\infty} \mathbb{P}\left[Y_1 + \cdots + Y_{n-1} \geq \frac{n}{2}\right] \\ &= \lim_{n\to\infty} \mathbb{P}\left[\frac{Y_1 + \cdots + Y_{n-1}}{n-1} \geq \frac{n}{2n-2}\right] \\ &= 1 - \lim_{n\to\infty} \mathbb{P}\left[\frac{Y_1 + \cdots + Y_{n-1}}{n-1} < \frac{n}{2n-2}\right] \\ {\scriptstyle \frac{n}{2n-2} \leq 2/3 - 0.01 \text{ for large } n} \quad &\geq 1 - \lim_{n\to\infty} \mathbb{P}\left[\frac{Y_1 + \cdots + Y_{n-1}}{n-1} < \frac{2}{3} - 0.01\right] \\ &= 1 \end{aligned}$$

[5] If Y_i are i.i.d random variables with $\mathbb{E}[Y_i] = \mu < \infty$ then for all $\varepsilon > 0$, $\lim_{m\to\infty} \mathbb{P}\left[\left|\frac{Y_1+\cdots+Y_m}{m} - \mu\right| > \varepsilon\right] = 0$.

On the other hand, all voters delegate to the center of the star, so $P_M(G) = 4/5$ when, say, M is the greedy local mechanism. Therefore $\text{gain}(M, G) \xrightarrow[n\to\infty]{} -1/5$ so M does not satisfy DNH.

Theorem 3.2. *For any $\alpha \in [0, 1)$, no local mechanism satisfies both PG and DNH.*

The crucial observation is that a local mechanism must exhibit the same behavior on all voters, which is captured by the following lemma.

Lemma 3.3. *Let M be a local mechanism that satisfies PG. Then there are $k, m, \rho > 0$ such that if a voter approves of k of his m neighbours (including, perhaps, themselves), the probability that the voter delegates (to someone other than themselves) is exactly ρ.*

Proof. Suppose M satisfies PG with gain $\gamma > 0$. Then there is a graph G such that $P_M(G) > P_D(G)$, so M must allow for some delegations on G, and specifically there is a voter i which delegates with nonzero probability. Let m be the number of neighbours i has, k be the number of approved neighbours, and ρ be the probability that i delegates. Since M is a local mechanism, we are done. □

The theorem is proved by constructing a graph that fools the mechanism into delegating many votes to a small amount of voters, which will cause it to fail as in 3.1.

Let M be a local mechanism that satisfies PG with delegation threshold $\alpha > 0$. Let (k, m, ρ) from lemma 3.3. A k-center m-uniform star is a graph with k center and a certain amount of outer vertices (at least $m - k$). Each outer vertex is connected to all k center vertices, and exactly $m - k$ leaves. The center vertices have no outgoing edges. Denote by n_c, p_c and n_o, p_o the number and competency of center and outer vertices, respectively. We will also add n_d entirely disconnected vertices of competency p_d. For an illustration, see

We then set the parameters so that the direct mechanism votes correctly with high probability, whereas if all centers vote incorrectly (which happens w.p $(1 - p_c)^{n_c}$), delegations to the center occur with high enough probability causing M to fail. The proof itself involves choosing the parameters to make this happen. While most of work is technical and involves that the construction is legal (e.g that n_d is nonnegative) which isn't too enlightening in our humble opinion, we will give the following high-level observation.

Fix $N \in \mathbb{N}$, we construct a graph with at least N vertices such that $\text{gain}(M, G) \approx (1 - p_c)^{n_c}$ for fixed p_c and n_c, which means that M does not satisfy DNH. The construction itself is parameterized by $\delta > 0$ and we will show that $\text{gain}(M, G_\delta) \xrightarrow[\delta\to 0]{} (1 - p_c)^{n_c}$. In the construction, we will have

$$\begin{aligned}
\alpha' &:= \frac{1+\alpha}{2} \\
n_c &:= k \\
p_c &:= \frac{1+\alpha'}{2} \\
n_l &:= \frac{Nm}{\alpha\delta} \\
p_l &:= \frac{1-\alpha'}{2}
\end{aligned}$$

notice that the noninteger values for n_l can be ignored, as we may restrict δ such that $1/\delta$ divides Nm/α. The choices for n_d and p_d are more involved, and are made to give D the advantage over M (so, p_d must be not too large and not too small).

4 Possibility for non-local mechanisms (an explicit construction)[KMP18]

As seen in theorem 3.2, liquid democracy might concentrate delegations to a small group of voters, so that if that small group errs then the election fails. A local mechanism has no way of knowing how delegations are spread, so cannot avoid this problem. But a non-local mechanism that avoids this problem exists, and works exactly by preventing individual voters from amassing too much weight. In words, takes as input a graph G and a cap $C : \mathbb{N} \to \mathbb{N}$, and iteratively selects the most approved voter and delegates $C(n-1)$ voters to it (so that its weight is $C(n)$). The resulting graph has only direct votes and immediate delegations (but no transitive ones). A formal description of `GreedyCap` can be found in 1.

While it is not hard to believe that `GreedyCap` satisfies PG (and is shown below), it's not clear why it Does No Harm. Indeed, without any further assumptions, that is not the case, as demonstrated in the following example.

Example 4.1. Let $n = 2k + 1$. Our network G_n is composed of k vertices of competence 1, $k - 1$ vertices of competence 0 and the remaining two vertices, v_1 and v_2, have competence 1/3 and 2/3. The only edge is (v_1, v_2). For `GreedyCap` to be distinct from D, we need $C \not\equiv 1$. Even taking $C \equiv 2$, `GreedyCap` delegates from v_1 to v_2 so that the success probability is 2/3 (as the entire election depends on v_2's choice), whereas $P_D(G) = 1 - \frac{1}{3} \cdot \frac{2}{3} = 7/9$ as in a direct democracy, either v_1 or v_2 can tip the scales. Thus the gain is $-1/9$.

The problem in the above example is that it takes a specific graph on which `GreedyCap` fails (the one with two vertices of unequal competencies), and makes `GreedyCap` fail asymptotically by adding "fake" vertices (indeed, all of the networks constructed are "equivalent" in some sense). Adding the assumption that there are no "fake" vertices, we obtain that `GreedyCap` satisfies DNH. Formally, we assume that there is $\beta \in (0, 1/2)$ such that $\text{Image}(p) \subseteq [\beta, 1-\beta]$ and that $\alpha \in (0, 1-2\beta)$.

Claim 4.2. `GreedyCap` satisfies PG.

Proof. Let G_n be a graph composed of pairs of (completely) connected nodes, one of competence β and the other of competence $1 - \beta$. It is clear that the $P_D(G_n) = 1/2$, whereas $P_{\texttt{GreedyCap}}(G_n) \xrightarrow[n\to\infty]{} 1$ from Condorcet's Jury Theorem (or similary to 3.1) as we are left with $n/2$ vertices of weight 2 and competence $1 - \beta > 1/2$. □

The proof that `GreedyCap` satisfies DNH is much more involved and, in our humble opinion, not enlightening.

5 The FAT angle on Liquid Democracy

Now that we've acquainted ourselves with liquid democracy and studied it rigorously, we can ask meaningful questions about liquid democracy from the FAT perspective.

- As promised, liquid democracy encourages *accountability*: After each vote, it is possible to change one's delegation.
- The question of *transparency* is more delicate: In general, a democratic vote shouldn't be entirely transparent, as lack of secrecy enables vote buying or extortion.
  - LiquidFeedback, a notable implementation of liquid democracy,[6] shows each user's vote weight but not specific delegations.
  - Nimrod Talmon and Ehud Shapiro suggest to allow participants to be either *public* or *private*: Public voters may receive delegations but have their voting history made public, whereas private voters may not.

[6] Used by several Pirate Parties (Germany, Austria, Switzerland, Italy, and Brazil) and some local branches of the Italian Five Star Movement

Part II
The Fluid Mechanics of Liquid Democracy[GKMP18]

6 The preferential delegation model

We consider the following adaptation to *preferential attachment* [BA99].

Definition 6.1. The *preferential delegation* model is characterized by three parameters: $d \in (0,1)$, $k \in \mathbb{N}$, and $\gamma \geq 0$.

At time t, a participant is added to the system. Then, w.p $1-d$ they vote directly. If they do not vote directly, they sample k nodes (who were already in the system) and choose to delegate to one of them (how? A delegation mechanism!), where each node j is sampled w.p proportional to $(\text{indeg}(j)+1)^{\gamma}$ (where indeg indicates the number of delegations).

A demonstration of *the power of choice* principle, we find that for $k=1$ super-voters (voters with a large amount of delegations) are always bound to emerge, where if we allows $k \geq 2$, then even greedy weight minimization allows us to keep the voting weight extremely low.

Theorem 6.2. *When $k=1$,$\gamma=0$, for any $d \in (0,1)$, the maximum of any voter at time t is $\Omega\left(t^{\beta}\right)$ with high probability, where β depends only on d. "With high probability" meaning that the probability that this event happens converges to 1 as $t \to \infty$.*

We will prove a weaker version of the above, in which the weight is high only in expectation.

Theorem 6.3. *When $k=1$,$\gamma=0$, for any $d \in (0,1)$, the expected weight of any voter at time t is $\Omega\left(t^{d}\right)$.*

The proof is delegated to section §A.

Finally, we quote the upper bound for maximal weight, when using the greedy algorithm when $k \geq 2$. The greedy algorithm works as follows: The new participant needs to decide whether to delegate to a or to b. Obtain voters a^* and b^* such that a delegates transitively to a^* and b delegates transitively to b^*. The new participant delegates to a iff the weight of a^* is less than the weight of b^*.

Theorem 6.4. *When $k=2$, $\gamma=0$, for any $d \in (0,1)$ the maximum weight of any voter at time t is $\log\log(t)+\Theta(1)$ with high probability.*

7 Minimizing the maximal weight is NP-hard to approximate

Definition 7.1. Consider a finite directed graph $G=(N,E)$ with voters $V \subseteq \text{sinks}(G)$.

$G'=(N',E')$ with voters V is a *delegation* graph if

- It is acyclic.
- Its sinks are exactly V.
- Every vertex in $N' \setminus V$ has outdegree one.

In such a graph, the *weight* of vertex n is defined as

$$w_{G'}(v) \coloneqq 1 + \sum_{(m,n)\in E} w_{G'}(m)$$

and w is finite since (N,E) is finite and acyclic.

Given G with $V \subseteq \text{sinks}(G)$ (not necessarily a delegation graph), a *delegation subgraph* is a subgraph $G' = (N', E')$ such that G' with V is a delegation graph. A delegation subgraph $G' = (N', E')$ is *of maximal size* if for each subgraph $G'' = (N'', E'')$ with $|N''| > |N'|$, G'' with V is *not* a delegation graph.[7]

The problem of interest is the MinMaxWeight problem: Given a directed graph $G = (N, E)$ with voters V, find a delegation subgraph (N', E') of maximal size that minimzes the maximal weight of the voting nodes. That is,

$$\text{find} \quad G^* \in \text{argmin}_{G' \text{ del. subg. of maximal size}} \max_{v \in V} w_{G'}(v)$$

It turns out that MinMaxWeight is NP-Hard to approximate.[8]

Theorem 7.2. *It is NP-hard to approximate* MinMaxWeight *to a factor of* $\frac{1}{2}\log|V^*|$, *even with the promise that each node has outdegree at most 2.*

The reduction from the problem of MinMaxCongestion of confluent flows. In short, a flow on a graph is *confluent* if each node has at most one outgoing edge with positive flow, and the *congestion* of a node is the incoming flow to the node (plus one).[9]

In fact, MinMaxCongestion and MinMaxWeight are closely tied, and an approximation algorithm to one immediately yields an approximation algorithm to the other. Thank to [CKLRSV07], we have a $1 + \log|V|$ approximation algorithm for this problem.

8 The FAT angle on Liquid Democracy revisited

Now that we've enriched our model we can consider more angles.

- If we use a nonlocal mechanism when $k = 2$, if the mechanism is entirely transparent then strategic voting (or delegation specification) is easier. For example, if we use the greedy mechanism, one can guaranatee that their vote is always delegated to voter a by specifying its two possible delegations as a and a voter with maximal weight. (If a is the voter with maximal delegations then we're in good shape anyways)
- So $k \geq 2$ is the way to go. Instead of having the mechanism decide how to delegate, we could have allowed splittable votes (all proofs for k analysis would work the same). But this is less accountable and transparent: To understand the outcome of their votes, participants might have to keep track of an exponential number of voters, whereas in the current model (where the mechanism decides only a single delegation) they only need to keep track of one voter.

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[7] These are the most interesting delegation subgraphs, as they are feasible while ignoring the minimal number of nodes.

[8] Some might argue that it is NP-Hard to define.

[9] This problem is NP-Hard by a reduction from the following problem: Given a directed graph and two pairs of nodes $(s, t), (s', t')$, find disjoint paths from s to t and from s' to t'.

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A Proof of 6.3

Denote by $w_t(i)$ the weight of node i at time t. Since $\mathbb{E}[\max_j w_t(j)] \geq \mathbb{E}[w_t(1)]$ is suffices to lower-bound $\mathbb{E}[w_t(1)]$.

Let D_i be the indicator random variable of the event that voter i trasnitively delegates to voter 1, and let $W_t := w_t(1) = \sum_{i=1}^{t} D_i$. It suffices to prove that $\mathbb{E}(W_t) = \Theta(t^d)$.

Observe that $\mathbb{E}[W_t]$ satisfies the following recurrences:

$$\mathbb{E}[W_1] = 1 \tag{A.1}$$

$$\mathbb{E}[W_{t+1}] = \left(1 + \frac{d}{t}\right)\mathbb{E}[W_t] \tag{A.2}$$

equation (A.1) is immediate. To prove equation (A.2), linearity of expectation gives us

$$\mathbb{E}[W_{t+1}] = \mathbb{E}[W_t] + \mathbb{E}[D_{t+1}] = \mathbb{E}[W_t] + \mathbb{P}[D_{t+1}]$$

so we should show that $\mathbb{P}[D_{t+1}] = \frac{d}{t}\mathbb{E}[W_t]$. Let Δ_{t+1}^{i} be the indicator of the event that in time $t+1$, the new voter decided to delegate and chose voter i which delegates to voter 1. Then $D_{t+1} = \bigsqcup_{i=1}^{t} \Delta_{t+1}^{i}$, and all coin tosses are independent,

$$\mathbb{P}\left[\Delta_{t+1}^{i}\right] = \underbrace{d}_{\text{new voter chose to delegate}} \cdot \underbrace{1/t}_{\text{new voter chose } i} \cdot \underbrace{\mathbb{P}[D_i]}_{i \text{ delegated to } 1}$$

therefore

$$\mathbb{P}[D_{t+1}] = \sum_{i=1}^{t} \mathbb{P}\left[\Delta_{t+1}^{i}\right] = \frac{d}{t}\sum_{i=1}^{t}\mathbb{P}[D_t] = \frac{d}{t}\mathbb{E}[W_t]$$

One can check that constraints equation (A.1) and equation (A.2) allow for at most one solution, and we shall show that a solution is obtained and is

$$\mathbb{E}[W_t] = \frac{\Gamma(t+d)}{\Gamma(d+1)\cdot\Gamma(t)}$$

where $\Gamma(z) = \int_{z=o}^{\infty} x^{z-1}e^{-x}\mathrm{dx}$ is Legendre's extension of the factorial function to real inputs (so $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}$). It satisfies equation (A.1), and as for equation (A.2), using properties of the Gamma function (one can think of the properties $(n+1)\cdot n! = (n+1)!$, though it is also straightforward to verify[10]),

$$\left(1+\frac{d}{t}\right)\frac{\Gamma(t+d)}{\Gamma(d+1)\cdot\Gamma(t)} = \frac{t+d}{t}\cdot\frac{\Gamma(t+d)}{\Gamma(d+1)\cdot\Gamma(t)} = \frac{(t+d)\Gamma(t+d)}{\Gamma(d+1)\cdot t\Gamma(t)} = \frac{\Gamma(t+d+1)}{\Gamma(d+1)\Gamma(t+1)}$$

To bound $\mathbb{E}[W_t]$ we use Gautschi's inequality:

[10] Hint: $\int x^{z-1}e^{-x}(x-z) = -e^{-x}x^z$

Fact (Gautschi's inequality)**.** *For any $x > 0$ and $s \in (0,1)$,*

$$x^{1-s} < \frac{\Gamma(x+1)}{\Gamma(x+s)} < (x+1)^{1-s}$$

Taking $x = t$ and $s = d$, and taking the reciprocal of all sides while reversing the relation, we have

$$(t+1)^{d-1} \leq \frac{\Gamma(t+d)}{\Gamma(t+1)} \leq t^{d-1}$$

Multiplying by t we have

$$t \cdot (t+1)^{d-1} \leq \frac{t \cdot \Gamma(t+d)}{\Gamma(t+1)} = \frac{\Gamma(t+d)}{\Gamma(t)} \leq t^{d-1} \cdot t = t^d$$

and dividing by $\Gamma(d+1)$ we get

$$\frac{t \cdot (t+1)^{d-1}}{\Gamma(d+1)} \leq \frac{\Gamma(t+d)}{\Gamma(d+1) \cdot \Gamma(t)} \leq \frac{t^d}{\Gamma(d+1)}$$

Since d is constant in our analysis, showing that the LHS and RHS are asymptotically the same will give us that $\mathbb{E}[W_t] = \frac{\Gamma(t+d)}{\Gamma(d+1)\cdot\Gamma(t)} = \Theta\left(t^d / \Gamma(d+1)\right) = \Theta\left(t^d\right)$ as required. Indeed,

$$\frac{t \cdot (t+1)^{d-1}}{\Gamma(d+1)} \Big/ \frac{t^d}{\Gamma(d+1)} = \frac{t \cdot (t+1)^{d-1}}{t^d} = \frac{(t+1)^{d-1}}{t^{d-1}} = \left(1 + \frac{1}{t}\right)^{d-1} \xrightarrow[t\to\infty]{} 1$$

Q.E.D

B Figures

Figure B.1: k-center m-uniform star[KMP18]

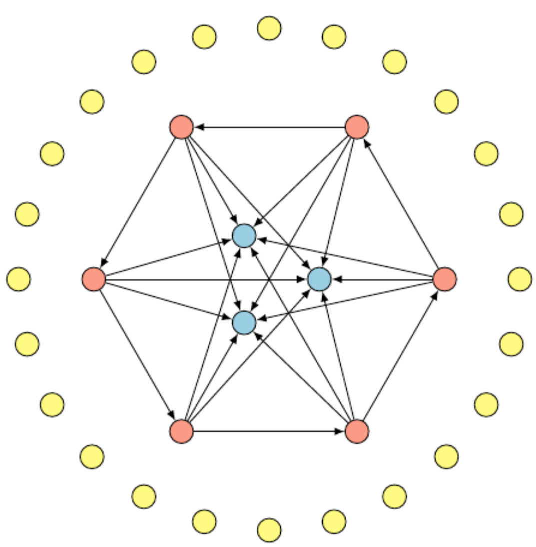

Figure 1: Graph G for $n_\ell = 6$ leaves (shown in red), $n_c = 3$ centers (shown in blue), $n_d = 24$ disconnected vertices (shown in yellow), and $m = 4$.

Algorithm 1 `GreedyCap`[GKMP18]

input: labeled graph G, cap $C : \mathbb{N} \to \mathbb{N}$

$V' \leftarrow V$
while $V' \neq \emptyset$ **do**
    let $i \in \text{argmax}_{j \in V'} |A_G(j) \cap V'|$
    $J \leftarrow A_G(i) \cap V'$
    if $|J| \leq C(n) - 1$ **then**
        $J' \leftarrow J$
    else
        let $J' \subseteq J$ such that $|J'| = C(n) - 1$
    end if
    vertices in J' delegate to i
    $V' \leftarrow V' \setminus (\{i\} \cup \{J'\})$
end while