Approximation from shift-invariant refinable spaces

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Refinement equations

\[ \hat{\Phi}(2 \cdot) = P \hat{\Phi} \]
Refinement equations

Compactly supported solutions

\[(D^\alpha \hat{\Phi}(0))_{|\alpha| \leq N}\]
determine \(\Phi\)
- Refinement equations
- Compactly supported solutions
- Appr. orders of SI spaces

\[ \Phi \subset W_2^s \text{ generates a SI space} \]

Q: find its appr. power
Outline

- Refinement equations
- Compactly supported solutions
- Appr. orders of SI spaces
- Appr. orders of smooth refinable functions

Smoothness + refinability \( \implies \) approximation power
Solutions to the same refinement equation are combined

- Refinement equations
- Compactly supported solutions
- Appr. orders of SI spaces
- Appr. orders of smooth refinable functions
- Coherent appr. orders
\[ v^*(2 \cdot P) - \delta_{l,0}v^* \] has a zero or order \( k \) at \( \pi l, \ l \in \{0, 1\}^d \), while \( v(0) \neq 0 \)

- Refinement equations
- Compactly supported solutions
- Appr. orders of SI spaces
- Appr. orders of smooth refinable functions
- Coherent appr. orders
- Condition \( Z_k \) and sum rules
Polynomials are reproduced by all solutions in a coherent way.

- Refinement equations
- Compactly supported solutions
- Appr. orders of SI spaces
- Appr. orders of smooth refinable functions
- Coherent appr. orders
- Condition $Z_k$ and sum rules
- Coherent polynomial reproduction
Refinement equations

Vector refinement equation

\[ \hat{\Phi}(2\cdot) = P\hat{\Phi}. \]

\( P \), a square matrix-valued \( 2\pi \)-periodic measurable function, is a refinement (matrix) mask,
\( \Phi \), a solution, is a refinable vector.
Vector refinement equation

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\( \Phi \), a solution, is a refi nable vector.

The space of all tempered distributional solutions \( \Phi \) is generally \textit{infinite-dimensional}, but

\[ R(P) := \text{the space of compactly supported solutions} \]

is always \textit{finite-dimensional}. 
Result [Jia, Jiang, Shen]. \( P \) a trig. polynomial,

\[
N := \max\{n : 2^n \in \sigma(P(0))\},
\]

\[
\mathcal{Z}_N := \{\alpha \in \mathbb{Z}_d^\perp : |\alpha| \leq N\}.
\]

The map

\[
\Phi \mapsto ((D^\alpha \hat{\Phi})(0))_{\alpha \in \mathcal{Z}_N}
\]

is then a bijection between the space \( R(P) \) and the kernel \( \ker L \) of the map

\[
L : \mathbb{C}^r \times \mathcal{Z}_N \to \mathbb{C}^r \times \mathcal{Z}_N :
\]

\[
(w_\alpha) \mapsto (2^{||\alpha||}w_\alpha - \sum_{0 \leq \beta \leq \alpha} (D^{\alpha-\beta}P)(0) w_\beta), \ \alpha \in \mathcal{Z}_N.
\]
Theorem. Suppose there are matrices $T$ and $\tilde{P}$ s.t.

(i) $T$ is analytic and invertible around the origin,

(ii) $\tilde{P}$ is a trig. polynomial,

(iii) $T(2^{\cdot})P - \tilde{PT} = O(| \cdot |^{N+1})$,

(iv) $\tilde{P}$ is block diagonal to order $N+1$ around 0 and the spectrum of each block evaluated at zero intersects the set $\{2^j : j = 0, \ldots, N\}$ at $\leq 1$ point.

Let $\Phi$ be in $R(P)$, and assume that each entry of $\hat{\Phi}$ has a zero of order $l$ at the origin. Then

$$\Phi = \sum_{j=l}^{N} p_j(D)\Phi_j, \quad \Phi_j \in R(P/2^j), \quad \hat{\Phi_j}(0) \neq 0,$$

and $p_j$ a homogeneous polynomial of degree $j$, $j = l, \ldots, N$. 
More about the space $\mathcal{R}(P)$

**Fact.** The ‘layer’ decomposition of the previous theorem may not exist.

**Example.** Let $d = 2$ and let $P$ be s.t.

$$P(0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & 4 \end{bmatrix},$$

$$(D^{(0,1)} P)(0) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$(D^{(1,0)} P)(0) = 0.$$
$F$ a space of functions over $\mathbb{R}^d$. $S \subset F$ is a shift-invariant (SI) space if

$$f \in S \implies f(\cdot - \alpha) \in S, \quad \text{all } \alpha \in (h)\mathbb{Z}^d.$$
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- A principal shift-invariant (PSI) space $S_\phi$ is the closure of

$$\text{span}[\phi(\cdot - j) : j \in \mathbb{Z}^d]$$

in the topology of $F$. 

SI spaces
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- A principal shift-invariant (PSI) space $S_\phi$ is the closure of

\[ \text{span}[\phi(\cdot - j) : j \in \mathbb{Z}^d] \]

in the topology of $F$.

- A finitely generated shift-invariant (FSI) space $S_\Phi$ is the closure of

\[ \sum_{\phi \in \Phi} S_\phi \]

in $F$, with $\Phi$ a finite subset of $F$. 

Approximation from shift-invariant refinable spaces – p.7/22
Approximation order

Sobolev space $W^s_2(\mathbb{R}^d)$: tempered distributions $f$ with $\hat{f}$ locally in $L_2(\mathbb{R}^d)$ and

$$\|f\|^2_{W^s_2(\mathbb{R}^d)} := \int_{\mathbb{R}^d} (1 + | \cdot |)^{2s} |\hat{f}|^2 < \infty.$$
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$$\|f\|_{W^s_2(\mathbb{R}^d)}^2 := \int_{\mathbb{R}^d} (1 + | \cdot |)^{2s} |\hat{f}|^2 < \infty.$$  

A ladder $S := (S^h := S^h(W^s_2))_{h>0}$ of SI spaces provides approximation order $k$, $k > s$, in $W^s_2(\mathbb{R}^d)$ if, for every $f \in W^k_2(\mathbb{R}^d)$,

$$\text{dist}_s(f, S^h) := \inf_{g \in S^h} \|f - g\|_{W^s_2(\mathbb{R}^d)} \leq C h^{k-s} \|f\|_{W^k_2(\mathbb{R}^d)},$$

with constant $C$ independent of $f$ and $h$. 
Characterization of appr. order

**Theorem.** An FSI stationary ladder \( (S^h := S^h(W_2^s)) \), with \( S^h = S_{\Phi}(\cdot/h) \), \( \Phi \subset W_2^s \), provides approximation order \( k > 0 \) if and only if there exists a neighborhood \( \Omega \) of 0 such that the function

\[
\mathcal{M}_{\Phi,s} : \omega \mapsto \frac{1}{|\omega|^{2k-2s}} \inf_{v \in \mathcal{C}_\Phi} \frac{v^* G^0_{\Phi,s}(\omega) v}{v^* G_{\Phi,s}(\omega) v}
\]

lies in \( L_\infty(\Omega) \). Here

\[
G_{\Phi,s} := \sum_{\alpha \in 2\pi \mathbb{Z}^d} \hat{\Phi}(\cdot + \alpha) \hat{\Phi}^*(\cdot + \alpha) |\cdot + \alpha|^{2s},
\]

\[
G^0_{\Phi,s} := \sum_{\alpha \in 2\pi \mathbb{Z}^d \setminus 0} \hat{\Phi}(\cdot + \alpha) \hat{\Phi}^*(\cdot + \alpha) |\cdot + \alpha|^{2s}
\]

(the **Gramian** and the truncated **Gramian**).
ψ ∈ S is a superfunction for $S$ if

$$\text{appr.order } (S_ψ) = \text{appr.order } (S).$$
Superfunctions

\[ \psi \in S \text{ is a superfunction for } S \text{ if} \]

\[ \text{appr.order} (S_\psi) = \text{appr.order} (S). \]

**Theorem.** Any FSI space \( S_\Phi \subset W^S_2(\mathbb{R}^d) \) contains a superfunction.
\( \psi \in S \) is a superfunction for \( S \) if

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\text{appr.order}(S_\psi) = \text{appr.order}(S).
\]

**Theorem.** Any FSI space \( S_\Phi \subset W_2^S(\mathbb{R}^d) \) contains a superfunction.

A superfunction for an FSI space \( S_\Phi \) is **good** if it is nondegenerate:

- \( |\hat{\psi}| \) is bounded away from 0 in a nbhd of 0,
- and **finitely spanned** by the shifts of \( \Phi \):

\[
\hat{\psi} = \tau^* \hat{\Phi}, \text{ with } \tau \text{ a trigonometric polynomial}.
\]
**Theorem.** If $S_{\phi} \subset W^s_{2}(\mathbb{R}^d)$ provides approximation order $k$, then

$$\hat{\phi}(\cdot + \alpha) = O(|\cdot|^k), \quad \text{all} \quad \alpha \in 2\pi \mathbb{Z}^d \setminus 0.$$
Polynomial reproduction

**Theorem.** If $S_\phi \subset W_2^s(\mathbb{R}^d)$ provides appr. order $k$ and $\phi$ is compactly supported, then

$$\phi *' \Pi_{<k} \subseteq \Pi_{<k}.$$ 

Here $*' \text{ is the semi-discrete convolution}$

$$g*' : f \mapsto \sum_{j \in \mathbb{Z}^d} g(\cdot - j)f(j),$$ 

$\Pi := \Pi(\mathbb{R}^d)$ is all $d$-variate polynomials, and $\Pi_{<k} := \{p \in \Pi : \deg p < k\}$. Moreover, if $\hat{\phi}(0) \neq 0$, then $\phi *' \Pi_{<k} = \Pi_{<k}$.
Theorem. Suppose $s \leq 0$, $\Phi \subset W_2^s$ is refinable, there exists a compact set $A$ s.t. $A \cap 2A$ has measure zero and $\bigcup_{m=-\infty}^{0} A/2^m$ contains a nbhd of 0, and some function $f \in S_\Phi(W_2^s)$ satisfies

- $|\hat{f}|$ is bounded above and away from zero on $A$;
- the numbers
  $$\lambda_m := \| \sum_{\alpha \in 2^m(2\pi\mathbb{Z}^d \setminus 0)} |\hat{f}(\cdot + \alpha)|^2 \|_{L_\infty(A)}, \ m \in \mathbb{Z}_+,$$
  decay as $\lambda_m = O(2^{-2mk})$, for some positive $k$.

Then $S_\Phi(W_2^s)$ provides approximation order $k$. 
Coherent appr. orders

To analyze:

\[ \omega \mapsto \inf_{v \in \mathcal{C}^{\Phi}} \frac{v^* G_{\Phi,s}^0(\omega)v}{v^* G_{\Phi,s}(\omega)v}. \]

Difficult already in \( L_2 \) since the Gramian \( G_{\Phi} \) of each solution \( \Phi \) is not invertible at zero if the refinement equation has multiple solutions.
Coherent appr. orders

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\[ \omega \mapsto \inf_{v \in \mathcal{C}_\Phi} \frac{v^* G_\Phi^0_s (\omega) v}{v^* G_\Phi^0_s (\omega) v}. \]

Difficult already in \( L_2 \) since the Gramian \( G_\Phi \) of each solution \( \Phi \) is not invertible at zero if the refinement equation has multiple solutions.

Result [Jiang, Shen]. Let \( \Phi \subset L_2 \) be a compactly supported refinable vector with Gramian \( G_\Phi \). If \( G_\Phi(0) \) is invertible, then the spectral radius \( \rho(P(0)) \) of \( P(0) \) is equal to 1, 1 is the only eigenvalue on the unit circle, and 1 is a simple eigenvalue.
Coherent appr. orders

Instead, analize this:

\[ \omega \mapsto \inf_{v \in C^\Phi} \frac{v^* G^0_{R(P),s} v}{v^* G_{R(P),s}(\omega) v}, \]

where \( G_{R(P),s} := \sum_{j=1}^{n} G_{\Phi_j,s} \) (combined Gramian),

\( G^0_{R(P),s} := \sum_{j=1}^{n} G^0_{\Phi_j,s} \) (truncated combined Gramian),

\( (\Phi_j) \) is a basis for \( R(P) \). We say that \( R(P) \) provides coherent appr. order \( k \) if there exists a nbhd \( \Omega \) of 0 such that the function \( M_{P,s,k} : \)

\[ \omega \mapsto \frac{1}{|\omega|^{2k-2s}} \inf_{v} \frac{v^* G^0_{R(P),s}(\omega) v}{v^* G_{R(P),s}(\omega) v} \] belongs to \( L_\infty(\Omega) \).
Universal supervectors

A vector $v$ that realizes coherent appr. order is a universal supervector. It is a regular universal supervector if

$$\frac{v^*G_{R(P),s}v}{v^*v} \sim | \cdot |^{2s}, \quad \frac{v^*G^0_{R(P),s}v}{v^*v} = O(| \cdot |^{2k}).$$

**Theorem.** Let $R(P)$ provide coherent appr. order $k$ in $W^s_2(\mathbb{R}^d)$.

- Let $S_P \subset W^s_2(\mathbb{R}^d)$ be the SI space generated by $R(P)$. Then $S_P$ is an FSI space and provides appr. order $k$. 
Let $\nu$ be a regular universal supervector of order $k$ bounded in a nbhd of 0. Then, for any $\Phi \in R(P)$,

(i) $\nu^* G_{\Phi,s} v = O(| \cdot |^{2k})$ around 0. The function $\psi$ defined by $\hat{\psi} := \nu^* \hat{\Phi}$ satisfies the Strang-Fix conditions of order $k$.

(ii) If $|\nu^* \hat{\Phi}| \geq c > 0$ a.e. in a nbhd of 0, then $S_\Phi$ provides appr. order $k$. Moreover, with $\psi \in S_\Phi$ defined by $\hat{\psi} := \nu^* \hat{\Phi}$, the PSI space $S_\psi$ already provides that appr. order.
Condition $Z_k$ and sum rules

Condition $Z_k$. $v^*(2\cdot)P - \delta_{l,0}v^*$ has a zero or order $k$ at each $\pi l$, $l \in \{0, 1\}^d$, while $v(0) \neq 0$. 
**Condition $Z_k$ and sum rules**

**Condition $Z_k$.** $v^*(2\cdot)P - \delta_{l,0}v^*$ has a zero or order $k$ at each $\pi l$, $l \in \{0, 1\}^d$, while $v(0) \neq 0$.

**Sum rules (variant I).**

$$\sum_{\sigma \in \mathbb{Z}^d} \sum_{\gamma \in \mathbb{Z}^d} v^*_{\sigma-\gamma} P_{l+2\sigma} q(l + 2\gamma) = 2^{-d} \sum_{\gamma \in \mathbb{Z}^d} v^*_{-\gamma} q(\gamma), \ l \in E, \ q \in \Pi_{<k}.$$
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Sum rules (variant II).

$$\sum_{\beta \leq \alpha} 2^{\lvert \alpha - \beta \rvert} (v^{\alpha-\beta})^*(D^\beta P)(\pi l) = \delta_{l,0}(v^\alpha)^*, \ l \in E, \ |\alpha| < k.$$
Condition $Z_k$ and sum rules

**Condition $Z_k$.** $v^*(2\cdot) P - \delta_{l,0} v^*$ has a zero or order $k$ at each $\pi l$, $l \in \{0, 1\}^d$, while $v(0) \neq 0$.

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**Sum rules (variant II).**

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\sum_{\beta \leq \alpha} 2^{||\alpha - \beta||} (v^{\alpha - \beta})^* (D^\beta P)(\pi l) = \delta_{l,0} (v^\alpha)^*, \; l \in E, \; |\alpha| < k.
$$

**Result.** Condition $Z_k \iff$ either of the sum rules.
Theorem. Let $P$ be a trig. polynomial mask, let $R(P) \subset W^s_2$, and let $v$ be a trig. polynomial vector. Suppose that $P$ and $v$ satisfy Condition $Z_k$ for some $k > 0$. Then:

- For each $\Phi \in R(P)$, the function $\psi$ defined by $\hat{\psi} := v^*\hat{\Phi}$ satisfies the Strang-Fix conditions of order $k$, and $\hat{\psi} - \hat{\psi}(0) = O(|\cdot|^k)$. Consequently, if $v^*(0)\hat{\Phi}(0) \neq 0$, then $S_{\Phi}(W^s_2)$ provides appr. order $k$, and $\psi \in S_{\Phi}(W^s_2)$ is a corresponding superfunction.

- If $v^*(0)\hat{\Phi}(0) \neq 0$ for some $\Phi \in R(P)$, then $R(P)$ provides coherent appr. order $k$, and $v$ is a corresponding universal regular supervector.
Theorem. Let $P$ be an $r \times r$ trig. polynomial mask. Suppose that $R(P) \subset L_2$ and that the combined Gramian $G_{R(P)}$ satisfies some technical assumptions, is smooth around each $l \in E$ and is boundedly invertible around each $l \in E$. If $P(0) = I$, TFAE:

(a) $P$ satisfies Condition $Z_k$ with some vector $v$.

(b) There exists a regular universal supervector $v$ of order $k$ for the space $R(P)$.

In addition, a regular universal supervector $v$ of order $k$ can be always chosen so that, for every $\Phi \in R(P)$,

$$v^* \hat{\Phi} - (v^* \hat{\Phi})(0) = O(|l| \cdot |k|).$$
Theorem. Let \( P \) be a refinement mask such that \( R(P) \subset W^s_2(\mathbb{R}^d) \). Let \( k > 0 \) and let \( v \) be a trig. polynomial such that one of the following holds:

- \( v \) satisfies Condition \( Z_k \).
- \( v \) and \( P \) satisfy either version of the sum rules
- \( v \) is a regular universal supervector of order \( k \).

Let \( \tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_r) \) be the sequence of the Fourier coefficients of \( v^* \), and let \( \Phi = (\phi_1, \ldots, \phi_r)' \in R(P) \). Then the map \( T_\Phi \) maps \( \Pi_{<k} \) into itself:

\[
T_\Phi : q \mapsto \sum_{i=1}^r \phi_i *' (\tilde{v}_i *' q) =: \Phi *' (\tilde{v} *' q)
\]

The map is surjective iff \( v^*(0)\widehat{\Phi}(0) \neq 0 \).
Post Scriptum

O. H. et Amos Ron, Approximation Orders of Shift-Invariant Subspaces of $W_2^s(\mathbb{R}^d)$, JAT 132/1 (2005), 97–148.

Thanks for your attention!