Bits and pieces of the nonnegative inverse eigenvalue problem

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Outline

- Nonnegative inverse eigenvalue problem (NIEP)
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• Necessary conditions
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- Functions that preserve nonnegativity of matrices
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- Related problems
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- Functions that preserve nonnegativity of matrices
- Preserving subclasses of nonnegative matrices
- Related problems
- Open problems
Given an $n$-tuple of complex numbers

$$\Lambda := (\lambda_1, \lambda_2, \ldots, \lambda_n),$$

does there exist a nonnegative matrix $A$ with

$$\sigma(A) = \Lambda?$$
NIEP

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$$\sigma(A) = \Lambda?$$

Contributions by:

Perron, Frobenius, Wielandt, Kellogg, Dmitriev, Dynkin, Suleimanova, Boyle, Handelman, Friedland, Laffey, Schneider, Neumann, Johnson, Chu, London, Loewy, and many more
Necessary conditions

- $\overline{\Lambda} = \Lambda$, conjugation
- $\max |\Lambda| \in \Lambda$, spectral radius
- $s_k(\Lambda) := \sum_{j=1}^{n} \lambda_j^k \geq 0, k \in \mathbb{N}$. moments

Moments $\implies$ conjugation & spectral radius conditions [Friedland, 1978].
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Moments $\iff$ conjugation & spectral radius conditions [Friedland, 1978].

Boyle & Handelman [1991]:

Any $n$-tuple $\Lambda$ satisfying basic conditions (= nonnegativity of moments) can be augmented by sufficiently many zeros so that the resulting $n$-tuple $\tilde{\Lambda} = (\Lambda, 0)$ will be realizable.
Connection to $M$-matrices

- sign pattern

\[
\begin{bmatrix}
+ & - & - & \cdots & - \\
- & + & - & \cdots & - \\
- & - & + & \cdots & - \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
- & - & - & \cdots & +
\end{bmatrix}
\]
Connection to $M$-matrices

- **sign pattern**

\[
\begin{bmatrix}
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- & + & - & \cdots & - \\
- & - & + & \cdots & - \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
- & - & - & \cdots & +
\end{bmatrix}
\]

- **nonnegativity of principal minors**

\[A[\alpha] \geq 0, \quad \text{all } \alpha\]
Connection to $M$-matrices

- $A$ is an $M$-matrix iff $A = rI - B$ where $B \geq 0$ and $r \geq \rho(B)$.

If $A$ is a nonsingular $M$-matrix, then $A^{-1}$ is nonnegative.

Open problem: characterize inverses of $M$-matrices.
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NIEP: further conditions


\[ s_k^m(\Lambda) \leq n^{m-1}s_{km}(\Lambda), \quad k, m \in \mathbb{N}. \]
**NIEP: further conditions**


  \[ s_k^m(\Lambda) \leq n^{m-1}s_{km}(\Lambda), \quad k, m \in \mathbb{N}. \]

- **Newton’s inequalities.**

  \( \Lambda \) is the spectrum of a nonnegative matrix \( \iff \)

  \[ \mathcal{M} := r(1, 1, \ldots, 1) - \Lambda, \quad r \geq \max |\Lambda|, \]

  is the spectrum of an \( M \)-matrix. Then

  \[ c_k^2(\mathcal{M}) \geq c_{k-1}(\mathcal{M})c_{k+1}(\mathcal{M}), \quad k = 1, \ldots, n, \]

  where \( c_k(\mathcal{M}) \) are normalized coefficients of the polynomial with roots \( \mathcal{M} \).
Newton’s inequalities

**Theorem.** Let $A$ be similar to an $M$- or inverse $M$-matrix. Then the normalized coefficients

$$c_j(A) := \sum_{\#\alpha = j} A[\alpha]/\binom{n}{j}, \quad j = 0, \ldots, n,$$

of its characteristic polynomial satisfy Newton’s inequalities

$$c_j^2(A) \geq c_{j-1}(A)c_{j+1}(A), \quad j = 1, \ldots, n - 1.$$
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These are **spectral conditions**.
More on Newton’s inequalities

**Corollary.** If $A$ is similar to an $M$- or an inverse $M$-matrix, then

$$c_j(A)^{1/j} \geq c_k(A)^{1/k} \quad \text{whenever } j \leq k.$$

In particular,

$$c_1(A) \geq c_j(A)^{1/j} \quad \text{for } j \geq 1.$$
Main question

For $\Lambda$ to be a solution to NIEP, it must satisfy moments, JLL, and Newton. Suppose $f$ is an entire function mapping nonnegative matrices of order $n$ into themselves:

$$A \geq 0 \iff f(A) \geq 0.$$ 

Then $f(\Lambda)$ must satisfy moments, JLL, and Newton. Thus we obtain many new inequalities necessary to solve NIEP.
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**Question:** Characterize all entire functions that leave invariant the cone of nonnegative matrices of order $n$. 

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Preliminary observations

Denote the class of entire functions preserving nonnegativity of matrices of order $n$ by $\mathcal{F}_n$ and its restriction to polynomials by $\mathcal{P}_n$.

- $\mathcal{F}_n$ contains all functions with nonnegative Taylor coefficients;
- $\mathcal{F}_n$ is closed under addition, multiplication, and composition;
- $\mathcal{P}_n$ is a union of proper cones obtained by restricting to polynomials of degree bounded by a fixed positive integer. The extreme directions of these cones may be of interest.
Easier questions

Which functions leave invariant these subclasses of nonnegative matrices of order $n$?

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- $ND_n$: nonnegative symmetric nonnegative definite matrices;
- $S_n$: nonnegative symmetric matrices;
- $C_n$: nonnegative circulant matrices;
- $U_n$ ($L_n$): nonnegative upper/lower triangular matrices;
Easier questions

Which functions leave invariant these subclasses of nonnegative matrices of order $n$?

- $ND_n$: nonnegative symmetric nonnegative definite matrices;
- $S_n$: nonnegative symmetric matrices;
- $C_n$: nonnegative circulant matrices;
- $U_n (L_n)$: nonnegative upper/lower triangular matrices;
- $BU_n (BL_n)$: nonnegative block upper/lower triangular matrices.
Preserving the class $ND_n$

All continuous functions that leave $ND_n$ invariant were characterized by Micchelli and Willoughby.

**Result** [Micchelli & Willoughby 1979]. A function $f$ continuous on $\mathbb{R}_+$ leaves invariant the class $ND_n$ of nonnegative definite entrywise nonnegative symmetric matrices of order $n$ iff all the divided differences of $f$ of order up to $n$ are nonnegative over $\mathbb{R}_+$:

$$f[x_1, \ldots, x_k] \geq 0 \quad x_1, \ldots, x_k \geq 0, \quad k = 1, \ldots, n.$$
Remark on the MW result

The result is not strong enough to characterize matrices preserving the class \( S_n \).

Example.

\[
f(x) = 1 + x + \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{4}x^4
\]

satisfies the Micchelli-Willoughby conditions of order 2, but it maps the matrix

\[
\begin{bmatrix}
0 & r \\
r & 0
\end{bmatrix}, \quad r \text{ sufficiently large},
\]

to a matrix with negative elements.
Preserving the class $U_n/L_n$

**Theorem.** A function $f$ continuous on $\mathbb{R}_+$ leaves invariant the class $U_n (L_n)$ of nonnegative upper-(lower-) triangular matrices of order $n$ iff all the divided differences of $f$ of order up to $n$ are nonnegative over $\mathbb{R}_+$.

**Proof:** is based on an explicit formula due to Schmitt and Stafney.
Preserving the class $U_n/L_n$

**Theorem.** A function $f$ continuous on $\mathbb{IR}_+$ leaves invariant the class $U_n$ ($L_n$) of nonnegative upper-(lower-) triangular matrices of order $n$ iff all the divided differences of $f$ of order up to $n$ are nonnegative over $\mathbb{IR}_+$.

Proof: is based on an explicit formula due to Schmitt and Stafney.

**Result** [Schmitt 1979, Stafney 1978].

\[
f(A)_{ij} = \begin{cases} 
f(a_{ii}) \\ 
\sum a_{ii_1} \cdots a_{i_k j} f(a_{ii}, a_{i_1 i_1}, \ldots, a_{i_k i_k}, a_{jj}) \\ 0 \end{cases}
\]
Preserving the class $C_n$

**Theorem.** An entire function $f$ maps the set $C_n$ of nonnegative circulant matrices of order $n$ into itself iff

$$
\sum_{k=0}^{n-1} \omega^{-lk} f(\sum_{j=0}^{n-1} \omega^{jk} a_j) \geq 0 \quad \text{whenever } a_j \geq 0,
$$

$$
j = 0, \ldots, n - 1, \quad \omega := e^{2\pi i/n}.
$$

Proof: is based on an explicit formula connecting eigenvalues of a circulant matrix to its elements.
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**Theorem.** An entire function $f$ maps the set $C_n$ of nonnegative circulant matrices of order $n$ into itself iff

$$\sum_{k=0}^{n-1} \omega^{-lk} f(\sum_{j=0}^{n-1} \omega^{jk} a_j) \geq 0 \quad \text{whenever } a_j \geq 0, \quad j = 0, \ldots, n - 1, \quad \omega := e^{2\pi i/n}.$$

Proof: is based on an explicit formula connecting eigenvalues of a circulant matrix to its elements.

**Result.** The eigenvalues of a circulant matrix with the first row $(a_0, \ldots, a_{n-1})$ are given by

$$\sum_{k=0}^{n-1} a_k \omega^{lk}, \quad l = 0, \ldots, n - 1.$$
Further results

- Preserving nonnegative block upper/lower triangular matrices ($BU_n/BL_n$).
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- Complete characterization for small values of $n$.

\[
n = 2 \quad f(x + y) - f(x - y) \geq 0, \quad x, y \geq 0
\]

\[
(x + y)f(x - y) + (y - x)f(x + y) \geq 0, \quad y \geq x \geq 0.
\]
Further results

Preserving nonnegative symmetric matrices ($S_n$) [joint work with G. Bharali].
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Preserving nonnegative symmetric matrices ($S_n$) [joint work with G. Bharali].

**Theorem.** An even function $x \mapsto f(x^2)$ leaves $S_n$ invariant iff all the divided differences of $f$ of order up to $n$ are nonnegative over $\mathbb{IR}_+$:

$$f[x_1, \ldots, x_k] \geq 0 \quad x_1, \ldots, x_k \geq 0, \quad k = 1, \ldots, n.$$  

The same condition is necessary and sufficient for an odd function $x \mapsto xf(x^2)$ to leave $S_n$ invariant.
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The same condition is necessary and sufficient for an odd function $x \mapsto xf(x^2)$ to leave $S_n$ invariant.

The class of functions preserving $S_n$ is **not the sum** of its even and odd subclasses.
Anti-bidiagonal matrix NIEP

**Theorem.** A real $n$-tuple $\Lambda$ can be realized as the spectrum of a symmetric anti-bidiagonal matrix $A$ if and only if $\Lambda = (\lambda_1, \ldots, \lambda_n)$ where

$$\lambda_1 > -\lambda_2 > \lambda_3 > \cdots > (-1)^{n-1}\lambda_n > 0.$$ 

The realizing matrix is necessarily unique.
Sketch of proof

An anti-bidiagonal matrix with two positive anti-diagonals is sign-regular with signature sequence

\[ 1, -1, -1, 1, 1, \cdots, (-1)^{\left\lfloor \frac{n-1}{2} \right\rfloor}. \]

Invoke a result by Gantmacher and Krein.
Sketch of proof

\[ \rightarrow \] An anti-bidiagonal matrix with two positive anti-diagonals is sign-regular with signature sequence

\[ 1, -1, -1, 1, 1, \cdots, (-1)^{n-1/2}. \]

Invoke a result by Gantmacher and Krein.
\[ \leftarrow \] Reduce to the same problem for Jacobi matrices

\[
\begin{bmatrix}
a_1 & a_2 & \cdots & 0 & 0 \\
a_2 & 0 & \cdots & 0 & 0 \\
& & & & \\
& & & & \\
0 & 0 & \cdots & 0 & a_n \\
0 & 0 & \cdots & a_n & 0
\end{bmatrix}
\]
Sketch of proof

“Reverse” the 3-term recurrence relation to arrive at

\[ q_n(\lambda) = (\lambda - a_1)q_{n-1}(\lambda) - a_2^2 q_{n-2}(\lambda), \]
\[ q_{n-j}(\lambda) = \lambda q_{n-j-1}(\lambda) - a_{j+2}^2 q_{n-j-2}(\lambda), \quad j = 1, \ldots, n - 2, \]
\[ q_0(\lambda) = 1, \quad q_1(\lambda) = \lambda. \]

With \( q_n \) given, find \( q_{n-1} \) and \( q_{n-2} \) from parity considerations. Prove root interlacing for \( q_n \) and \( q_{n-1} \). The rest is easy (using interlacing repeatedly).
Application

Corollary. Let $\mathcal{M}$ be a real positive $n$-tuple. Then there exists a Jacobi matrix that realizes $\mathcal{M}$ as its spectrum and has a symmetric anti-bidiagonal square root of the form

$$A = \begin{bmatrix}
0 & 0 & \cdots & 0 & a_n \\
0 & 0 & \cdots & a_{n-2} & a_{n-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & a_{n-2} & \cdots & 0 & 0 \\
a_n & a_{n-1} & \cdots & 0 & 0 \\
\end{bmatrix}, \quad a_1, \ldots, a_n > 0.$$
Open problems

- Can Newton’s inequalities be used to prove inequalities conjectured by Boyle & Handelman and Goldberger & Neumann?

- Can Newton’s inequalities be used to characterize nonnegative matrices that are inverses of M-matrices?

- What about functions preserving other subclasses of nonnegative matrices, e.g., Toeplitz or Hankel?

- Can the cone theory and the semigroup theory be used to characterize functions that preserve nonnegativity of matrices?
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Papers

- $M$-matrices satisfy Newton’s inequalities [Proceedings of AMS 2005]
- The inverse eigenvalue problem for symmetric anti-bidiagonal matrices [LAA 200?]
- Functions preserving nonnegativity of matrices, with G. Bharali [coming soon]

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