Solutions to the mock Putnam test.

1. Expanding the given expression, we get

$$2(2a^4+2b^4+4a^3b+6a^2b^2+4ab^3) = 4(a^4+b^4+(ab)^2+2a^2(ab)+2b^2(ab)+2a^2b^2) = 4(a^2+b^2+ab)^2.$$

2. The condition is equivalent to

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 0. \tag{1}$$

Since (1) is invariant under any translation $z \mapsto z - z_0$ and rotation $z \mapsto z_0 z$, $z_0 \in \mathbb{C} \setminus \{0\}$, we may assume without loss of generality that $a = 0$ and that b is a positive real number. Note that c cannot be real, since then the left-hand side of (1) would be strictly positive. Writing c in its polar form $\varrho e^{i\theta}$, we obtain

$$b^2 + \varrho^2(\cos 2\theta + i \sin 2\theta) = b\varrho(\cos\theta + i\sin\theta). \tag{2}$$

Comparing imaginary parts of both sides and using the double angle formula, we get

$$(2\varrho\cos\theta - b)\sin\theta = 0.$$

We know already that c is not real, so $\sin\theta \neq 0$. Hence $\cos\theta = \frac{b}{2\varrho}$ and $\cos 2\theta = \frac{b^2}{2\varrho^2} - 1$. Now comparing the real parts in (2), we get $b^2 + \frac{b^2}{2} - \varrho^2 = \frac{b^2}{2}$. Hence $b = r$ and $\cos\theta = \frac{1}{2}$, so the triangle formed by points a, b, c is equilateral. The argument also shows that situation to be the unique one where (1) holds.

3. If a polynomial identity holds in $\mathbb{R}$, it also olds over $\mathbb{C}$. Hence $p(z^2) = p(z)p(z-1)$ for all $z \in \mathbb{C}$. Since p is nonconstant, it has roots in $\mathbb{C}$. First note that 0 cannot be a root of p, since then we would have

$$\begin{aligned} p(1^2) = p(1)p(1-1) &= 0, \\ p(2^2) = p(2)p(2-1) &= 0, \\ p(5^2) = p(5)p(5-1) &= 0, \quad \text{etc.,} \end{aligned}$$

so p would have infinitely many (nonnegative real) roots. So, $p(0) \neq 0$.

Now, if a is a root of p, so are a^2, a^4, ..., a^{2^n}. Therefore this sequence must have a finite range and, since $a \neq 0$, a must be a root of unity. Note also that $(a+1)^2$ is a root of p as well, hence $a+1$ is also a root of unity. In particular, $|a| = |a+1| = 1$, which is possible only for $a = e^{\pm 2i\pi/3}$. Since p has real coefficients, both values, which are conjugates of each other, are (the only) roots of p and occur with the same multiplicity. Thus

$$p(x) = c((x - e^{2i\pi/3})(x - e^{-2i\pi/3}))^m = c(x^2+x+1)^m.$$

By the functional equation, $c = 1$, so $p(x) = (x^2+x+1)^m$ for some $m \in \mathbb{N}$.

Answer: $p(x) = (x^2+x+1)^m$, $m \in \mathbb{N}$.

4. If a function f is analytic, then $\frac{\partial f(z)}{\partial \overline{z}} = 0$ (this is in fact just a reformulation of the Cauchy-Riemann conditions). By the same token, $\frac{\partial \overline{f(z)}}{\partial z} = 0$. On the other hand, $\frac{\partial \overline{f(z)}}{\partial \overline{z}} = \overline{f'(z)}$. So,

$$\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}\,|f_j(z)|^2 = \frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}\,f_j(z)\overline{f_j(z)} = \frac{\partial}{\partial z}\,f_j(z)\overline{f_j'(z)} = f_j'(z)\overline{f_j'(z)} = |f_j'(z)|^2.$$

Differentiating the right-hand side, we get $\frac{\partial}{\partial z}\frac{\partial}{\partial \overline{z}}(az + b\overline{z} + c) = 0$, so

$$\sum_{j=1}^{n} |f_j'(z)|^2 = 0,$$

hence each f_j' is zero, and since the domain D is connected, each f_j is constant in D.

5. The set of all $0-1$ matrices has cardinality 2^{n^2}. From the definition of determinant,

$$(\det A)^2 = \left(\sum_{\sigma} \varepsilon(\sigma) \prod_{i=1}^{n} a_{i,\sigma(i)}\right)\left(\sum_{\tau} \varepsilon(\tau) \prod_{i=1}^{n} a_{i,\tau(i)}\right) = \sum_{\sigma,\tau} \varepsilon(\sigma)\varepsilon(\tau) \prod_{i=1}^{n} a_{i,\sigma(i)}a_{i,\tau(i)},$$

where $\varepsilon(\sigma)$ is ± 1 depending on whether σ is even or odd. This sum can be rewritten more simply if we denote the permutation $\sigma^{-1}\tau$ by π. Then $\varepsilon(\sigma)\varepsilon(\tau) = \varepsilon(\sigma^{-1})\varepsilon(\tau) = \varepsilon(\pi)$, and the formula becomes

$$(\det A)^2 = \sum_{\sigma,\pi} \varepsilon(\pi) \prod_{i=1}^{n} a_{i,\sigma(i)}a_{i,\sigma\pi(i)}.$$

Let $f(\pi)$ denote the number of indices fixed by π. Then, for each σ, the corresponding product has $2n - f(\pi)$ distinct factors, and there are $2^{n^2-2n+f(\pi)}$ matrices in which all these elements are equal to 1. So the expectation of each term is

$$E\left(\prod_{i=1}^{n} a_{i,\sigma(i)}a_{i,\sigma\pi(i)}\right) = \frac{2^{f(\pi)}}{2^{2n}}$$

and, by the linearity of expectation,

$$E((\det A)^2) = \frac{n!}{2^{2n}} \sum_{\pi} \varepsilon(\pi) 2^{f(\pi)}.$$

It remains to find the sum in the right-hand side. Notice that that sum is the value of a determinant itself, namely, the determinant of the $n \times n$ matrix C whose entries are

$$c_{i,j} = \begin{cases} 1 & i \neq j \\ 2 & i = j. \end{cases}$$

The value of $\det C$ is $n+1$, which can be seen by replacing its first row by the sum of all rows, factoring out $n+1$ and then subtracting the first row from every other row. So, the resulting expectation value is $E((\det A)^2) = (n+1)!/2^{2n}$.

Answer: $(n+1)!/2^{2n}$.