Quadratic reciprocity.

Definitions. An integer a satisfying $(a, m) = 1$ is called a *quadratic residue* modulo m if there exists a solution to the congruence $x^2 \equiv a \pmod{m}$. Otherwise a is a *quadratic nonresidue* modulo m. Let p be an odd prime. The *Legendre symbol* $\left(\frac{a}{p}\right)$ is defined as

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo } p, \\ 0 & \text{if } p|a. \end{cases}$$

Basic facts about the Legendre symbol. Let p be an odd prime. Then

1. $a \equiv b \pmod{p} \Longrightarrow \left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$,
2. $\left(\frac{a^2}{p}\right) = 1$ unless $p|a$,
3. $\left(\frac{a}{p}\right)\left(\frac{b}{p}\right) = \left(\frac{ab}{p}\right)$,
4. $\left(\frac{-1}{p}\right) = (-1)^{(p-1)/2}$,
5. $\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$.

Theorem [law of quadratic reciprocity]. Let p and q be distinct odd primes. Then

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\frac{q-1}{2}}.$$

Examples.

1. Prove that there are no integers x and y for which
$$x^2 + 3xy - 2y^2 = 122.$$
2. Show that there are no integers a, b for which $2b^2 + 3$ divides $a^2 - 2$.
3. Show that there are infinitely many primes of the form $3k + 1$.

Additional olympiad problems on number theory.

1. Show that the cube roots of three distinct primes cannot be three terms (not necessarily consecutive) of an arithmetic progression.
2. Let p and q be natural numbers such that
$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{1318} + \frac{1}{1319}.$$
Prove that p is divisible by 1979.

3. Let $s(n)$ denote the sum of all digits of n in decimal notation. Evaluate

$$s(s(s(4444^{4444}))).$$

4. Show that there is no natural number d that makes each of the numbers $2d-1$, $5d-1$, and $13d-1$ a perfect square.

5. Prove that

$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}.$$