Rational and irrational numbers.

Definition. A *rational number* is one that can be written in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$. In this representation, a and b can be chosen to be relatively prime. Then the fraction is said to be *in lowest terms.* The set of all rational numbers is denoted by $\mathbb{Q}$, the set of real numbers by $\mathbb{R}$. Real numbers that are not rational are called *irrational.*

Basic fact. A real number is rational if and only if its decimal expansion (or expansion in any base) is periodic.

Less trivial fact. Any number of the form $\sqrt[k]{n}$, where k and n are natural numbers larger than 1 and n is not a perfect kth power, is irrational.

Further definitions. A number that satisfies an equation of the form

$$a_0 + a_1 x + \cdots + a_n x^n = 0,$$

where a_0, a_1, ..., a_n are integers and $a_n \neq 0$, is called *algebraic.* A number that is not algebraic is called *transcendental.*

Famous transcendentality results. The numbers e and π are transcendental.

Rational root theorem. If a_0, a_1, ..., a_n are integers and $\frac{a}{b}$ is in lowest terms and is a root of the equation

$$a_0 + a_1 x + \cdots + a_n x^n = 0,$$

then $a|a_0$ and $b|a_n$.

Examples.

1. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.
2. Prove that there is no set of integers except 0, 0, 0 for which $m + n\sqrt{2} + p\sqrt{3} = 0$.
3. Suppose a real number α satisfies $\cos(\alpha\pi) = 1/3$. Prove that α is irrational.
4. Let $\frac{a}{b}$ (in lowest terms) represent a rational number in the interval $(0, 1)$. Prove that

$$\left|\frac{a}{b} - \frac{\sqrt{2}}{2}\right| > \frac{1}{4b^2}.$$

5. Prove the following classical fact: if $\alpha \in \mathbb{Q}$, then the sequence $(\{n\alpha\} : n \in \mathbb{N})$ is periodic; if $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then the sequence $(\{n\alpha\} : n \in \mathbb{N})$ is dense in $(0, 1)$. Here $\{x\}$ denotes the *fractional part* of x:

$$\{x\} := x - \lfloor x \rfloor.$$

6. Prove *Beatty's theorem:* If α and β are positive irrational numbers satisfying $1/\alpha + 1/\beta = 1$, then the sequiences $(\lfloor n\alpha \rfloor : n \in \mathbb{N})$ and $(\lfloor n\beta \rfloor : n \in \mathbb{N})$ together include every natural number exactly once.