Progressions and sums.

Definition. An *arithmetic progression* is a sequence (a_n) (finite or infinite) with the property that, for a fixed *common difference* d and all indices n, $a_{n+1} - a_n = d$. An equivalent property is $a_n = (a_{n-1} + a_{n+1})/2$.

Basic facts. The kth term of an arithmetic progression with initial term a_1 and common difference d is

$$a_k = a_1 + (k-1)d.$$

The sum of the first n terms of an arithmetic progression is

$$a_1 + a_2 + \cdots + a_n = n\left(\frac{a_1 + a_2}{2}\right).$$

Definition. A *geometric progression* is a sequence (a_n) (finite or infinite) with the property that, for a fixed *common ratio* r and all indices n, $a_{n+1} = ra_n$.

Basic facts. The kth term of a geometric progression with initial term a_1 and common ratio d is

$$a_k = a_1 r^{k-1}.$$

The sum of the first n terms of a geometric progression is

$$a_1 + a_2 + \cdots + a_n = \begin{cases} \frac{a_1(1-r^n)}{1-r} & \text{if } r \neq 1 \\ a_1 n & \text{if } r = 1. \end{cases}$$

An infinite geometric series with $|r| \geq 1$ diverges. The sum of an infinite geometric series with common ratio r, $|r| < 1$, and initial term a_1 is

$$\sum_{n=1}^{\infty} a_n = \frac{a_1}{1-r}.$$

Summing powers of integers. Consider

$$S_r(n) := \sum_{k=1}^{n} k^r.$$

Then

$$\begin{aligned} S_1(n) &= n(n+1)/2 \\ S_2(n) &= n(n+1)(2n+1)/6 \\ S_3(n) &= (n(n+1)/2)^2. \end{aligned}$$

Generally,

$$S_r(n) = \frac{B_{r+1}(n+1) - B_{r+1}(0)}{r+1},$$

where $B_m(x)$ is the mth *Bernoulli polynomial.* Bernoulli polynomials satisfy the relation $B_m(x+1) - B_m(x) = mx^{m-1}$.

Examples.

1. Let the set of natural numbers be partitioned into groups as follows:
$$1, (2, 3), (4, 5, 6), (7, 8, 9, 10), \ldots$$
Find the sum of the integers in the nth group.

2. Prove that the powers of 2 are the only positive integers that cannot be written as the sum of two or more consecutive integers.

3. Let (x_n) be a sequence of nonzero real numbers satisfying
$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}}$$
for $n = 3, 4, 5, \ldots$ Establish a necessary and sufficient condition on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

4. Suppose $a_1 = 2$ and $a_{k+1} = 3a_k + 1$ for all $k \geq 1$. Find a general formula for
$$a_1 + a_2 + \cdots + a_n.$$

5. Show that
$$\sin\left(\frac{\pi}{n}\right) + \sin\left(\frac{2\pi}{n}\right) + \cdots + \sin\left(\frac{(n-1)\pi}{n}\right) = \cot\left(\frac{\pi}{2n}\right).$$

6. Evaluate
$$\sum_{k=1}^{n} \frac{1}{(k+1)\sqrt{k} + k\sqrt{k+1}}.$$

7. Show that there are no four consecutive binomial coefficients
$$\binom{n}{k}, \binom{n}{k+1}, \binom{n}{k+2}, \binom{n}{k+3}$$
which are in arithmetic progression.