Polynomials.

Definitions. A *polynomial* in m variables is a function

$$\sum_{\alpha \in I} c(\alpha) x^{\alpha}$$

where I is a finite subset of $\mathbb{Z}_+^m$ of multi-indices $\alpha =:(\alpha(1), \ldots, \alpha(m))$, and the corresponding *monomials* are defined by $x^\alpha := x_1^{\alpha(1)} \cdots x_m^{\alpha(m)}$. Polynomials are treated as formal expressions in algebra and as functions on $\mathbb{R}^m$ or $\mathbb{C}^m$ in analysis. The (nonzero) numbers $c(\alpha)$ are called *coefficients* of p. The index α of the monomial occuring in p that is highest in a chosen monomial order determines the *degree* of p. In the univariate case, the degree is just the biggest power of the variable that occurs in p. In the multivariate setting, various orders are possible, so the same polynomial may have different degrees depending on the order chosen.

Univariate polynomials are by now well understood.

Fundamental theorem of algebra. Every nonzero univariate polynomial p of degree n with complex coefficients has exactly n roots $(a_j)_1^n$ in $\mathbb{C}$ and can be factored as

$$p(z) = a(z - a_1)(z - a_2) \cdots (z - a_n).$$

Uniqueness theorem. If p and q are univariate polynomials of degree at most n and $p(x_j) = q(x_j)$ for $j = 1, 2 \ldots, m$ where $x_1, x_2, \ldots, x_m$ are distinct complex numbers and $m > n$, then p and q are identical.

Theorem [division algorithm]. If f and g are univariate polynomials and g is not the zero polynomial, then there exist unique polynomials q and r such that

$$f(x) = q(x)g(x) + r(x)$$

where either r is the zero polynomial or $\deg r < \deg g$. The *quotient* q and the *remainder* r can be found by synthetic division.

Bezout's theorem. The remainder from the division of a polynomial $f(x)$ by $x - a$ is equal to $f(a)$.

Theorem [univariate polynomial interpolation]. For any sequence $(a_j)_1^n$ of complex numbers and a set of n distinct points $(x_j)_1^n$ from $\mathbb{C}$, there exists a unique polynomial p of degree at most $n - 1$ such that

$$p(x_j) = a_j, \qquad j = 1, \ldots, n.$$

Examples.

1. Find the remainder when $x^{81} + x^{49} + x^{25} + x^9 + x$ is divided by $x^3 - x$.
2. Let p be a nonconstant polynomial with integral coefficients. If $n(p)$ is the number of distinct integers k such that $(p(k))^2 = 1$, prove that $n(p) - \deg(p) \leq 2$ where $\deg(p)$ denotes the degree of the polynomial p.

3. Factor $(a+b+c)^3 - (a^3+b^3+c^3)$.
4. Find a if a and b are integers such that $x^2 - x - 1$ is a factor of $ax^{17} + bx^{16} + 1$.
5. Let $r \neq 0$ be given. Find the polynomial p of degree at most n that satisfies

$$p(j) = r^j, \qquad j = 0, \ldots, n.$$

6. Find the unique polynomial p of degree n that satisfies

$$p(j) = \frac{1}{1+j}, \qquad j = 0, \ldots, n.$$

   Hint: consider $(x+1)p(x) - 1$.
7. A polynomial p of degree 990 satisfies $p(k) = F_k$ for $k = 992, 993, \ldots, 1982$, where F_k denotes the kth Fibonacci number. Prove that $p(1983) = F_{1983} - 1$.

Further reading. Polynomial interpolation and divided differences by Carl de Boor, available at `http://www.cs.wisc.edu/~deboor/887` as pages pp. 48–53 of his class notes for Math 887, Approximation Theory.