Pigeonhole principle.

Basic principle. If n objects are placed in k boxes, then at least one of the boxes contains $\lceil n/k \rceil$ objects or more.

Theorem [Dirichlet]. Let α be an irrational number. Then there are infinitely many integer pairs (h, k) where $k > 0$ such that

$$\left|\alpha - \frac{h}{k}\right| < \frac{1}{k^2}. \tag{1}$$

Proof. Let Q be an arbitrary positive integer. First show that there exist integers h and k such that $1 \le k \le Q$ and

$$\left|\alpha - \frac{h}{k}\right| < \frac{1}{kQ}.$$

To this end, consider the "boxes"

$$B_k := [(k-1)/Q, k/Q), \qquad k = 1, \ldots, Q,$$

and the "objects" $\{q\alpha\}$, $q = 0, \ldots, Q$, where $\{x\}$ denotes, as usual, the fractional power of x. By the pigeonhole principle, one box contains at least two objects. This implies that $|\{q_1\alpha\} - \{q_2\alpha\}| < 1/Q$ for some $0 \le q_1 < q_2 \le Q$. Setting $h := \lfloor q_2\alpha \rfloor - \lfloor q_1\alpha \rfloor$ and $k := q_2 - q_1$, we get

$$\left|\alpha - \frac{h}{k}\right| < \frac{1}{kQ}.$$

Now, suppose there does not exist an infinite collection satisfying (1). Then, for some $\varepsilon > 0$ and all pairs (h, k) satisfying (1), we get $|\alpha - h/k| > \varepsilon$. But this implies

$$\left|\alpha - \frac{h}{k}\right| < \frac{1}{kQ} \le \min\{\frac{1}{k^2}, \frac{1}{Q}\} \le \min\{\frac{1}{k^2}, \varepsilon\} \text{ whenever } Q > 1/\varepsilon.$$

Contradiction!

Theorem [Erdös and Szekeres]. Every sequence of $(m-1)(n-1)+1$ distinct real numbers has either an increasing subsequence of length m or a decreasing subsequence of length n.

Proof. Place an element of the sequence in a box labeled r if the longest increasing subsequence beginning with that element has r terms. If there is no subsequence with m terms, we need only $m-1$ boxes. By the pigeonhole principle, the placement of $(m-1)(n-1)+1$ elements in these boxes yields a box with at least n elements. These n terms form a decreasing subsequence, since any two terms forming an increasing subsequence belong to different boxes.

To see that this result is best possible, consider the sequence of intervals

$$(I_1, \ldots I_{n-1}) \text{ where } I_k := [(n-1-k)(m-1)+1, (n-k)(m-1)].$$

This sequence consists of $(m-1)(n-1)$ distinct integers and has neither an increasing subsequence with m terms nor a decreasing subsequence with n terms.

Examples.

1. Given integers $1 \leq a_1, \ldots, a_m \leq n$ and $1 \leq b_1, \ldots, b_n \leq m$, show that there are integers p, q, r, s such that
$$a_p + a_{p+1} + \cdots a_q = b_r + b_{r+1} + \cdots + b_s.$$
2. There are n people at a party. Prove that there are two people such that of the remaining $n-2$ people, there are at least $\lfloor n/2 \rfloor - 1$ of them each of whom knows either both or neither of the two. Assume that "knowing" is a symmetric relation.
3. In a circular arrangement of zeros and ones, with n terms altogether, prove that if the number of ones exceeds $(k-1)n/k$, then there must be a string of k consecutive ones.
4. A given $m \times n$ matrix A of real numbers satisfies $a_{i1} \leq a_{i2} \leq \cdots \leq a_{in}$ for $i = 1, \ldots, m$. The elements in each column are then rearranged to obtain a new matrix B satisfying $b_{1j} \leq b_{2j} \leq \cdots \leq b_{mj}$ for all $j = 1, \ldots, n$. Prove that $b_{i1} \leq \cdots \leq b_{in}$ for all $i = 1, \ldots, m$.
5. Suppose that the squares of an $n \times n$ chessboard are labeled arbitrarily with numbers 1 through n^2. Prove: there are two adjacent squares whose labels differ by at least n.