Mock Putnam test, Mon, Oct 4th.

1. For any integers a and b, prove that

$$2(a^4 + b^4 + (a+b)^4)$$

is a perfect square.

2. Show that a necessary and sufficient condition for three points a, b, and c in the complex plane to form an equilateral triangle is that

$$a^2 + b^2 + c^2 = bc + ca + ab.$$

3. Find all nonconstant polynomials p with real coefficients such that

$$p(x^2) = p(x) \cdot p(x-1) \qquad \text{for all } x \in \mathbb{R}.$$

4. Suppose that functions f_j, $j = 1, \ldots, n$, are analytic in an open connected set $D \subseteq \mathbb{C}$ and satisfy the following condition

$$\sum_{j=1}^{n} |f_j(z)|^2 = az + b\bar{z} + c \qquad \text{for all } z \in S$$

for some constants a, b, $c \in \mathbb{C}$. Show that each f_j is a constant function in D (and so, *a posteriori*, $a = b = 0$).

5. Find the expected value of $(\det A)^2$ for a random $n \times n$ matrix A whose entries are independently distributed random variables taking values 0 and 1, each with probability 1/2.