1. For any integers $a$ and $b$, prove that
\[ 2(a^4 + b^4 + (a + b)^4) \]
is a perfect square.

2. Show that a necessary and sufficient condition for three points $a$, $b$, and $c$ in the complex plane to form an equilateral triangle is that
\[ a^2 + b^2 + c^2 = bc + ca + ab. \]

3. Find all nonconstant polynomials $p$ with real coefficients such that
\[ p(x^2) = p(x) \cdot p(x - 1) \quad \text{for all} \quad x \in \mathbb{R}. \]

4. Suppose that functions $f_j$, $j = 1, \ldots, n$, are analytic in an open connected set $D \subseteq \mathbb{C}$ and satisfy the following condition
\[ \sum_{j=1}^{n} |f_j(z)|^2 = az + b \bar{z} + c \quad \text{for all} \quad z \in S \]
for some constants $a, b, c \in \mathbb{C}$. Show that each $f_j$ is a constant function in $D$ (and so, a posteriori, $a = b = 0$).

5. Find the expected value of $(\det A)^2$ for a random $n \times n$ matrix $A$ whose entries are independently distributed random variables taking values 0 and 1, each with probability $1/2$. 