# Linear algebra in a nutshell.

**Definition.** A vector space $V$ over a field $F$ is endowed with two operations: addition, with respect to which $V$ is an abelian group, and multiplication by scalars from $F$, which satisfies associative and distributive laws. Precisely, the following axioms define a vector space:

- For all $x$, $y \in V$, $x + y \in V$.
- For all $x$, $y \in V$, $x + y = y + x$.
- For all $x$, $y$, $z \in V$, $x + (y + z) = (x + y) + z$.
- There exists a zero vector 0 such that $x + 0 = x$ for all $x \in V$.
- For each vector $x \in V$, there exists a vector $-x$ such that $x + (-x) = 0$.
- For all $x \in V$ and $\alpha \in F$, $\alpha x \in V$.
- For all $x \in V$ and the unit $1 \in F$, $1x = x$.
- For all $x \in V$ and $\alpha$, $\beta \in F$, $\alpha(\beta x) = (\alpha\beta)x$.
- For all $\alpha$, $\beta \in F$ and $x \in V$, $(\alpha + \beta)x = \alpha x + \beta x$.
- For all $\alpha \in F$ and $x$, $y \in V$, $\alpha(x + y) = \alpha x + \alpha y$.

**Definition.** A subset $S \subseteq V$ is said to *span* $V$ if every vector in $V$ can be expressed as a linear combination of vectors from $S$:

$$x = \sum_{v \in S} \alpha_v v, \qquad \alpha_v \in F.$$

A subset $S$ is said to be *linearly dependent* if the zero vector can be written as a nontrivial linear combination of vectors from $S$. Otherwise $S$ is called *linearly independent*. $S$ is a *basis* of $V$ is it is both linearly independent and spans $V$. The cardinality of any basis of $V$ is called its *dimension*. This number does not depend on the particular basis chosen.

**Linear maps.** A map $A$ between two vector spaces $A : V \to W$ is *linear* if $A(\alpha x + \beta y) = \alpha Ax + \beta Ay$ for all $x$, $y \in V$ and $\alpha$, $\beta \in F$. A linear map between finite-dimensional spaces $V$, $W$ is completely described by its matrix $(A_{ij})$ once a basis for $V$ and a basis for $W$ are chosen. Then $A_{ij}$ gives the $i$th coordinate, with respect to the $W$-basis, of the image under $A$ of the $j$th basis vector for $V$. A linear map $A : V \to V$ is described by a square matrix. Changes of basis correspond to *nonsingular,* i.e., invertible square matrices.

The *rank* of $A$ is the dimension of its range. The *nullity* of $A$ is the dimension of its kernel.

**Rank-nullity theorem.** Let $A : V \to W$ be a linear map. Then

$$\dim(\text{range}A) + \dim(\ker A) = \dim V.$$

**Canonical forms and invariants.** A linear map between two *different* vector spaces is completely characterized by its rank. Its matrix can be transformed into the *reduced* *row-echelon form* using a map $A \mapsto T^{-1}AS$ where $T$ and $S$ are nonsingular matrices corresponding to changes of bases on $W$ and $S$, respectively.

**Eigenvalues and eigenvectors.** Let $A : V \to V$ be a linear map. If $Ax = \lambda x$ for some nonzero vector $x$ and an arbitrary scalar $\lambda$, then $(\lambda, x)$ are said to be an *eigenpair,* $\lambda$ being an *eigenvalue* and $x$ being an *eigenvector.*

**Characteristic and minimal polynomials.** Let $A$ be a square matrix. The polynomial in $\lambda$

$$\mathrm{poly}(\lambda) := \det(\lambda I - A)$$

is called its *characteristic polynomial.*

**Theorem [Hamilton, Cayley].** The characteristic polynomial annihilates its matrix:

$$\mathrm{poly}(A) = 0.$$

The monic polynomial of minimal degree that annihilates a (square) matrix is called its *minimal* polynomial. It divides every polynomial that annihilates $A$.

**Jordan normal form.** A linear map $A$ from a space to itself is reducible to its *Jordan normal form.* The form is completely described by eigenvalues of $A$ and the sizes of the corresponding Jordan blocks. These form a complete set of invariants under *similarity:* $A \mapsto T^{-1}AT$ where $T$ is an arbitrary change of basis.

Both the characteristic and the minimal polynomial are easy to read off from the Jordan normal form of a matrix. The minimal polynomial of $A$ is a product of factors $(\lambda - \lambda_j)^{\alpha_j}$ where $(\lambda_j)$ are all distinct eigenvalues of $A$ and $\alpha_j$ is the maximal size of a Jordan block associated with $\lambda_j$. The characteristic polynomial is a product of factors $(\lambda - \lambda_j)^{\beta_j}$ where $(\lambda_j)$ are as above and $\beta_j$ are sums of sizes of all Jordan blocks associated with $\lambda_j$.

**Inner product spaces.** A vector space may be equipped with an additional multiplicative structure, namely., with an *inner* (or *scalar*) *product* $\langle \cdot, \cdot \rangle : V \times V \to F$. Over the field $F = \mathbb{C}$, an inner product satisfies the following axioms:

- For all $x$, $y \in V$, $\langle x, y \rangle = \overline{\langle y, x \rangle}$.
- For all $x \in V$, $x \cdot x \geq 0$; $\langle x, x \rangle = 0$ if and only if $x = 0$.
- For all $x$, $y$, $z \in V$: $\langle x, (\alpha y + \beta z) \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$.

The standard inner product on the space $\mathbb{C}^n$ is given by the formula

$$\langle x, y \rangle := x^* y = \sum_{j=1}^{n} \overline{x(j)} y(j).$$

A basis can be always chosen on an $n$-dimensional inner product vector space $V$ so that the inner product acts in the standard way with respect to that basis. The process is called the *Gram-Schmidt* orthogonalization.

**Linear maps and inner products.** A linear map that preserves an inner product in the sense that $\langle Ux, Uy \rangle = \langle x, y \rangle$ for all $x$, $y \in V$ is called *unitary* (with respect to that product).

A unitary matrix with respect to the standard inner product is characterized by the property

$$UU^* = I$$

where $I$ is the identity matrix of order $n$ and $U^*$ is the conjugate transpose of $U$. All eigenvalues of a unitary matrix lie on the unit circle.

A linear map is *self-adjoint* (or *Hermitian*) (with respect to an inner product) if $\langle Hx, Hy \rangle = \langle x, y \rangle$ for all $x$, $y \in V$. In the standard inner product, such maps are characterized by the fact

$$H = H^*.$$

All eigenvalues of a Hermitian matrix are real. A Hermitian matrix is *positive definite* if $x^* H x > 0$ for all $x \neq 0$. A positive definite Hermitian matrix has only positive eigenvalues.

**Theorem [Schur].** A *normal matrix,* i.e., a matrix with the property

$$NN^* = N^* N$$

is diagonalizable under unitary similarity

$$A \mapsto U^* A U, \qquad \text{where } UU^* = I.$$

Any matrix is upper- (or lower-) triangularizable under unitary similarity.

**Examples.**

1. Let $A$ be an $r \times r$ matrix with integer entries. Suppose that

   $$\sum_{j=1}^{r} a_{ij} = n, \qquad 1 \leq i \leq r.$$

   Prove that $n$ is a divisor of $\det A$.

2. Let $A$ be a linear map on a finite-dimensional vector space over a finite field. Show that if $A$ is invertible, then $A^n = I$ for some $n$.

3. Let $A$ and $B$ be two Hermitian matrices and let $A$ in addition be positive definite. Show that all eigenvalues of $AB$ are real.

4. Let $A$ and $B$ be two Hermitian matrices with eigenvalues in the intervals $[a_1, a_2]$ and $[b_1, b_2]$, respectively. Show that the eigenvalues of $A+B$ lie in the interval $[a_1+b_1, a_2+b_2]$.

5. Prove or disprove: There is a real $n \times n$ matrix $A$ such that

   $$A^2 + 2A + 5I = 0$$

   if and only if $n$ is even.

6. Prove or disprove: a square complex matrix is similar to its transpose.

7. Let $A$ be an $n \times n$ complex matrix. Show that

$$\lim_{k\to\infty} A^k = 0$$

if and only if all eigenvalues of $A$ have absolute value less than 1.

8. Let $A$ be a real $n \times n$ matrix satisfying

$$a_{ii} > 1 \text{ for all } i, \quad \sum_{i\neq j} a_{ij}^2 < 1.$$

Prove that $A$ is invertible.