Mathematical induction.

Basic principle. Let T be a set of natural numbers satisfying the following two conditions:

1. $1 \in T$,
2. if $n \in T$, then $n + 1 \in T$.

Then $T = \mathbb{N}$.

Condition 1 is referred to as the *induction base* and condition 2 as the induction step. Condition 2 may be replaced by

2^*. if $1, 2, \ldots n \in T$, then $n + 1 \in T$.

This version is sometimes called *strong induction*. Here is yet another equivalent version.

Well-ordering principle. Every nonempty subset of $\mathbb{N}$ has a smallest element.

Examples.

1. Prove that, for every natural number n,
$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}.$$
2. Prove that $n(n-1)(n+1)(3n+2)$ is divisible by 24.
3. Suppose that $a_1 = 3$ and $a_{n+1} = a_n(a_n + 2)$ for all $n \geq 1$. Find a formula for the nth term of this sequence.
4. Prove *Bernoulli's inequality:*
$$(1 + x)^n \geq 1 + nx$$
for every real number $x \geq -1$ and every natural number n.
5. Derive a formula for the sum $1^4 + 2^4 + \cdots + n^4$.
6. Given a $(2m+1) \times (2n+1)$ checkerboard where the four corner squares are black, show that if one removes any one red and two black squares, the remaining board can be covered with dominoes (1×2 rectangles).
7. Consider a set of $2n$ points in space, $n > 1$. Suppose they are joined by at least $n^2 + 1$ segments. Show that at least one triangle is formed. Show that for each n it is possible to have $2n$ points joined by n^2 segments without forming a triangle.