Inclusion-exclusion principle.

Inclusion-exclusion formula. Suppose $P_1, \ldots, P_n$ are subsets of S. Let

$$M_k := \sum_{|I|=k} M(\supseteq I) \quad \text{where} \quad M(\supseteq I) := |\cap_{i \in I} P_i|.$$

Then the number of elements of S that belong to precisely r of the sets P_j is given by

$$E_r = \sum_{k=r}^{n} (-1)^{k-r} \binom{k}{r} M_k. \tag{1}$$

Proof. Note first that

$$M_k = \sum_{r=k}^{n} \binom{n}{k} E_r$$

since an element belonging to exactly r of the P_j's contributes $\binom{r}{k}$ times to the sum defining M_k. Consider polynomials

$$M(x) := \sum_{k=0}^{n} M_k x^k, \qquad E(x) := \sum_{r=0}^{n} E_r x^r.$$

Then

$$M(x) = \sum_{k=0}^{n} \sum_{r=k}^{n} \binom{r}{k} E_r x^k = \sum_{r=0}^{n} E_r \sum_{k=0}^{r} \binom{r}{k} x^k = \sum_{r=0}^{n} E_r (x+1)^r = E(x+1).$$

It follows that

$$E(x) = M(x-1) = \sum_{k=0}^{n} M_k (x-1)^k = \sum_{k=0}^{n} M_k \sum_{r=0}^{k} \binom{k}{r} x^r (-1)^{k-r} = \sum_{r=0}^{n} x^r \sum_{k=r}^{n} (-1)^{k-r} \binom{k}{r} M_k,$$

from which we get (1).

Derangements. Let D_n denote the number of permutations of $[n]$ with no fixed points. Then

$$D_n = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Proof. There are $\binom{n}{k}$ choices for $I \subseteq [n]$ with $|I| = k$, and for each one there are $(n-k)!$ permutations of $[n]$ such that $\pi(i) = i$ for all $i \in I$. Thus $M_k = \binom{n}{k}(n-k)! = n!/k!$. So,

$$D_n = \sum_{k=0}^{n} (-1)^k M_k = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

More generally, the number of permutations of $[n]$ having exactly r fixed points is given by

$$E_r = \sum_{k=r}^{n} (-1)^{k-r} \binom{k}{r} \frac{n!}{k!} = \frac{n!}{r!} \sum_{j=0}^{n-r} \frac{(-1)^j}{j!} = \binom{n}{r} D_{n-r}.$$

Examples.

1. Show that the number of permutations π of $[n]$ such that $\pi(j+1) \neq \pi(j)+1$ for $j = 1, \ldots, n-1$ is $D_n + D_{n-1}$.

2. Prove that
$$\left\{ {n \atop k} \right\} = \frac{1}{k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (k-j)^n.$$

3. How many $m \times n$ $0-1$ matrices have no row or column consisting entirely of zeros?

4. Find the generating function
$$G_n := \sum_{\sigma \in S_n} x^{w(\sigma)}$$
where S_n denotes the set of all permutations of $[n]$ and $w(\sigma)$ is the number of fixed points of σ.

5. Using the inclusion-exclusion formula, show that
$$\sum_{U \subseteq S} (-1)^{|U|} \binom{m - \sigma(U)}{|S|} = \pi(S) \quad \text{for all } m \geq \sigma(S),$$
where $|U|$, $\sigma(U)$, and $\pi(U)$ denote the number of elements, the sum and the product, respectively, of a finite set U. (For the empty set, these numbers are 0, 0, 1.)