Homework # 9, due Nov 29th.

1. For what integer a does $x^2 - x + a$ divide $x^{13} + x + 90$?

2. Given five points in a plane, no three of which lie on a straight line, show that some four of these points form the vertices of a convex quadrilateral.

3. Let S be the set of all numbers of the form $2^m 3^n$, where m and n are integers, and let P be the set of all positive real numbers. Is S dense in P?

4. Let α and β be given positive real numbers, with $\alpha < \beta$. If two points are selected at random from a straight line segment of length β, what is the probability that the distance between them is at least α?

5. How many n-digit integers are there, consisting only of the digits 1 through 5, in which adjacent digits differ by exactly 1?

6. Show that the set of real numbers that satisfy the inequality

$$\sum_{k=1}^{70} \frac{k}{x-k} \geq \frac{5}{4}$$

is a union of disjoint intervals, the sum of whose lengths is 1988.

7. Prove that there is a constant K such that the following inequality holds for any sequence of positive numbers $a_1, a_2, a_3, \ldots$:

$$\sum_{n=1}^{\infty} \frac{n}{a_1 + a_2 + \cdots + a_n} \leq K \sum_{n=1}^{\infty} \frac{1}{a_n}.$$

8. Prove that

$$\lim_{n\to\infty} e^{n/4} n^{-(n+1)/2} (1^1 2^2 \cdots n^n)^{1/n} = 1.$$

9. Let P_1, P_2, $\ldots$ be a sequence of distinct points which is dense in the interval $(0,1)$. The points P_1, P_2, $\ldots$, P_{n-1} decompose the interval into n parts, and P_n decomposes one of these into two parts. Let a_n and b_n be the lengths of these two intervals. Prove that

$$\sum_{n=1}^{\infty} a_n b_n (a_n + b_n) = 1/3.$$

10. Given an infinite number of points in a plane, prove that if all the distances between them are integers, then the points are all on a straight line.

11. Let

$$\begin{matrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots \\ a_{2,1} & a_{2,2} & a_{2,3} & \cdots \\ a_{3,1} & a_{3,2} & a_{3,3} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

be a doubly infinite array of positive integers, and suppose each positive integer appears exactly eight times in the array. Prove that $a_{m,n} > mn$ for some pair of positive integers (m, n).

12. Let S be a set of three, not necessarily distinct, positive integers. Show that one can transform S into a set containing 0 by a finite number of applications of the following rule: Select two of the three integers, say x and y, where $x \leq y$ and replace them with $2x$ and $y - x$.