Homework # 8, due Nov 19th.

1. A square of side $2a$, lying always in the first quadrant of the XY plane, moves so that two consecutive vertices are always on the X- and Y-axes, respectively. Find the locus of the midpoint of the square.

2. Let g be a real function that has a continuous first derivative g' on $\mathbb{R}$. Suppose further that $g(0) = 0$ and $|g'(x)| \leq |g(x)|$ for all $x \in \mathbb{R}$. Prove that g vanishes identically.

3. Let (a_n) be a decreasing sequence of positive numbers with limit 0 such that

$$b_n := a_n - 2a_{n+1} + a_{n+2} \geq 0 \qquad \text{for all} \quad n \in \mathbb{N}.$$

Prove that $\sum_{n=1}^{\infty} nb_n = a_1$.

4. Does the series

$$\sum_{n=2}^{\infty} \frac{\cos(\ln \ln n)}{\ln n}$$

converge or diverge?

5. Given a sequence (a_n) satisfying the conditions

$$a_1 = 2, \quad a_{n+1} = a_n^2 - a_n + 1, \qquad n \in \mathbb{N},$$

prove 1) $(a_n, a_m) = 1$ whenever $m \neq n$, 2) $\sum_{n=1}^{\infty} 1/a_n = 1$.

6. Let a function f be defined by the equality

$$\cos 17x =: f(\cos x).$$

Show that then $\sin 17x = f(\sin x)$.

7. Let $x_1, x_2, \ldots, x_n$ be positive numbers. Show that

$$\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \frac{x_3}{x_4 + x_5} + \cdots + \frac{x_n}{x_1 + x_2} > (\sqrt{2} - 1)n.$$

8. Suppose that n points are given on the unit circle so that the product of the distances of any point of the circle from these points is not greater than 2. Prove that the points are the vertices of a regular n-gon.

9. A fair die bearing number 1 through 6 on its faces is thrown repeatedly until the running total first exceeds 12. What is the most likely total that will be obtained?

10. Prove that, for all $\alpha \in (0, \pi)$ and all $n \in \mathbb{N}$,

$$\sin \alpha + \frac{1}{2} \sin 2\alpha + \frac{1}{3} \sin 3\alpha + \cdots + \frac{1}{n} \sin n\alpha > 0.$$

11. Let n and k be given natural numbers, and let A be a set such that

$$|A| \leq \frac{n(n+1)}{k+1}.$$

For $j = 1, \ldots, n+1$, let A_j be sets of size n such that

$$|A_i \cap A_j| \leq k \qquad \text{for} \quad i \neq j, \qquad\qquad A = \cup_{j=1}^{n+1} A_j.$$

Determine the cardinality of A.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable, 2π-periodic even function such that

$$f''(x) + f(x) = \frac{1}{f(x + 3\pi/2)} \qquad \text{for all} \quad x \in \mathbb{R}.$$

Prove that f is $\pi/2$-periodic.