Homework # 7, due Nov 5th.

1. Which planes cut the surface $xy + xz + yz = 0$ in 1) circles, 2) parabolas?

2. Prove that every positive number has a multiple whose decimal representation involves all ten digits.

3. Prove that the sequence

$$\sqrt{7},\ \sqrt{7-\sqrt{7}},\ \sqrt{7-\sqrt{7+\sqrt{7}}},\ \sqrt{7-\sqrt{7+\sqrt{7-\sqrt{7}}}},\ \ldots$$

converges, and evaluate the limit.

4. In the Gregorian calendar:

   1. years not divisible by 4 are common years;
   2. years divisible by 4 but not by 100 are leap years;
   3. years divisible by 100 but not by 400 are common years;
   4. years divisible by 400 are leap years;
   5. a leap year contains 366 days, a common year 365.

Prove that the probability that Christmas falls on a Wednesday is not 1/7.

5. Prove that, for all $n \in \mathbb{N}$,

$$\frac{1}{n^2} + \left(\frac{1}{n} + \frac{1}{n-1}\right)^2 + \cdots + \left(\frac{1}{n} + \frac{1}{n-1} + \cdots + 1\right)^2 = 2n - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n}\right).$$

6. Let (a_n) be an arbitrary sequence of positive numbers. Show that

$$\limsup_{n\to\infty} \left(\frac{a_1 + a_{n+1}}{a_n}\right)^n \geq e.$$

7. Prove that in any arithmetic progression of natural numbers two numbers can always be found with equal sums of their digits.

8. Can one cover the whole plane, without overlaps, by squares with side lengths 1, 2, 4, 8, 16, etc. using each square at most 1) ten times? 2) once?

9. Suppose P and Q are $n \times n$ matrices such that $P^2 = P$, $Q^2 = Q$, and $I - (P+Q)$ is invertible. Show that P and Q have the same rank.

10. Let f be a function on $\mathbb{R}$ such that

$$\lim_{x \to \infty} \frac{f(x)}{e^x} = 1$$

and $|f''(x)| \leq c|f'(x)|$ for some constant c and all sufficiently large x. Prove that

$$\lim_{x \to \infty} \frac{f'(x)}{e^x} = 1.$$

11. Let $x_1, \ldots, x_n$ be n points in the unit square ($n > 1$). Let r_j be the distance of x_j to the nearest point other than itself. Prove the inequality

$$r_1^2 + \cdots + r_n^2 \leq 4.$$

12. Show that if $k \leq n/2$ and $\mathcal{F}$ is a family of $k \times k$ submatrices of an $n \times n$ matrix such that any two intersect, then

$$|\mathcal{F}| \leq \binom{n-1}{k-1}^2.$$