Homework # 6, due Fri, Oct 29th.

1. Write all values of i^{i} in the form $a + \mathrm{i}b$, $a, b \in \mathbb{R}$.

2. Find all positive integers that are within 250 of exactly 15 perfect squares.

3. Of the 1985 people attending a conference, no one spoke more than 5 languages and, in any subset of 3 people of those assembled, at least 2 spoke a common language. Prove that some language was spoken by at least 200 of the people at the conference.

4. Let C_1 and C_2 be circles whose centers are 10 units apart, and whose radii are 1 and 3. Find, with proof, the locus of all points M for which there exists points X on C_1 and Y on C_2 such that M is the midpoint of the line segment XY.

5. Solve

$$5^x 7^y + 4 = 3^z$$

over nonnegative integers (x, y, z).

6. Prove

$$\binom{n}{1} - 2\binom{n}{2} + 3\binom{n}{3} - \cdots + (-1)^{n-1} n \binom{n}{n} = \begin{cases} 0, & n \neq 1 \\ 1, & n = 1. \end{cases}$$

7. If $\sum_{m=-\infty}^{\infty} |a_m| < \infty$, then what can be said about the limit

$$\lim_{n\to\infty} \frac{1}{2n+1} \sum_{m=-\infty}^{\infty} |a_{m-n} + a_{m-n+1} + \cdots + a_{m+n}| \;?$$

8. Let $c \geq 0$ be a constant. Give a complete description, with proof, of the set of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) = f(x^2 + c)$ for all $x \in \mathbb{R}$.

9. Prove that every infinite sequence S of distinct positive integers contains either

(a) an infinite subsequence such that, for every pair of its elements, neither element divides the other, or

(b) an infinite subsequence such that, in every pair of elements, one always divides the other.

10. Let $A := (a_{i,j})$, $B := (b_{i,j})$ be $n \times n$ real symmetric *nonnegative definite* matrices, i.e.,

$$x'Ax \geq 0, \qquad x'Bx \geq 0 \qquad \text{for all } x \in \mathbb{R}^n.$$

Prove that the *Hadamard product* $C := (a_{i,j} b_{i,j})$ of A and B is also nonnegative definite.

11. Prove that if a_j $(j = 1, 2, 3, 4)$ are positive constants, $a_2 - a_4 > 2$ and $a_1 a_3 - a_2 > 2$, then the solution $(x(t), y(t))$ to the system of differential equations

$$\begin{aligned} \dot{x} &= a_1 - a_2 x + a_3 x y, \\ \dot{y} &= a_4 x - y - a_3 x y, \end{aligned}$$

with the initial conditions $x(0) = 0$, $y(0) \geq a_1$ is such that the function $x(t)$ has exactly one strict local maximum on the interval $[0, \infty)$.

12. Prove that an idempotent linear operator on a Hilbert space is self-adjoint if and only if it has norm 0 or 1.