Homework # 5, due Mon, Oct 18th.

1. Suppose that a sequence $(a_n)_{n\in\mathbb{N}}$ satisfies

$$0 < a_n \le a_{2n} + a_{2n+1} \qquad \text{for all } n \in \mathbb{N}.$$

Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

2. Find the minimum value of

$$\frac{(x+1/x)^6 - (x^6+1/x^6) - 2}{(x+1/x)^3 + (x^3+1/x^3)}$$

for $x > 0$.

3. Prove that in the sequence

$$2^1+1,\ 2^2+1,\ 2^4+1,\ 2^8+1,\ \ldots,\ 2^{2^n}+1,\ \ldots$$

all elements are pairwise relatively prime.

4. The lock on a safe consists of three wheels A, B, and C, each of which may be set in eight different positions. Due to a defect in the mechanism, the door will open when any two of the wheels are in the correct position. Thus anybody can open the safe in 64 tries by letting B run through all 8 position for *each* position of A. However, the safe can be always opened in far fewer tries than that. What is the minimum number of tries that is guaranteed to open the safe?

5. Let us color the integers $1, 2, \ldots, N$ with three colors so that each color is given to more than $N/4$ integers. Show that the equation $x = y + z$ has a solution where x, y, and z are of distinct colors.

6. Let A and B be $n \times n$ complex matrices. Prove that

$$|\mathrm{tr}(AB^*)|^2 \le \mathrm{tr}(AA^*)\mathrm{tr}(BB^*).$$

7. Evaluate

$$\int_{-\infty}^{\infty} \frac{\sin x}{x(x-\pi)} \mathrm{dx}.$$

8. Let $f(x) := (r_1 - x)(r_2 - x) \cdots (r_n - x)$. Show that

$$\begin{vmatrix} r_1 & a & a & \cdots & a \\ b & r_2 & a & \cdots & a \\ b & b & r_3 & \cdots & a \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b & b & b & \cdots & r_n \end{vmatrix} = \frac{af(b) - bf(a)}{a-b}.$$

9. Prove that the set of all linear combinations (with real coefficients) of the system of polynomials $(x^n + x^{n^2})_{n=0}^{\infty}$ is dense in $C[0,1]$.

10. Prove that the nth partial sum of the exponential series

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$$

has either no real roots or exactly one real root, depending on whether n is even or odd.

11. Let f be a twice differentiable real valued function on $[0, 2\pi]$, with $\int_0^{2\pi} f(x)\mathrm{dx} = 0 = f(2\pi) - f(0)$. Show that

$$\int_0^{2\pi} (f(x))^2 \mathrm{dx} \leq \int_0^{2\pi} (f'(x))^2 \mathrm{dx}.$$

12. Determine all real numbers x for which the following statement is true: the field $\mathbb{C}$ of complex numbers contains a proper subfield F such that adjoining x to F we get $\mathbb{C}$.