Homework # 4, due Fri, Oct 8st.

1. An unknown polynomial has remainder 5 when divided by $x-1$ and remainder -5 when divided by $x+1$. What is its remainder from the division by $(x-1)(x+1)$?

2. From a point D on the hypotenuse BC of a right triangle ABC, perpendiculars DE and DF are drawn to AC and AB, respectively. Determine the position of D for which EF has minimum length.

3. Let $m \in \mathbb{N}$. Show that the highest power of 2 that divides

$$\lfloor (1+\sqrt{3})^{2m+1} \rfloor$$

is 2^{m+1}.

4. Let g be a polynomial of dgree $2n$ satisfying

$$z^{2n}\overline{g}\left(\frac{1}{z}\right) = g(z).$$

How are the zeros of g distributed in the complex plane?

5. If $a+b+c=0$, prove that relation

$$\frac{a^5+b^5+c^5}{5} = \frac{a^3+b^3+c^3}{3} \cdot \frac{a^2+b^2+c^2}{2}.$$

6. Prove that

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}\right) = \ln 2.$$

7. Prove that if a function $f : \mathbb{R}^2 \to [0,1]$ is continuous and its average on every circle of radius 1 equals the function value at the center of the circle, then f is constant.

8. Prove that a continuous function from $\mathbb{R}$ to $\mathbb{R}$ that maps open sets to open sets must be monotonic.

9. Prove that the number of partitions of n in which all the even parts are distinct is the same as the number of partitions of n in which each part is repeated at most 3 times.

10. Five different numbers are drawn at random from $\{1,2,\ldots,n\}$, one at a time, without replacement. Show that the probability that the first three numbers drawn, as well as all five numbers, can be arranged to form an arithmetic progression is greater than

$$\frac{6}{(n-2)^3}.$$

11. Let p be a prime of the form $4n+1$. Prove that

$$\lfloor\sqrt{p}\rfloor + \lfloor\sqrt{2p}\rfloor + \cdots + \lfloor\sqrt{(p-1)p/4}\rfloor = \frac{p^2-1}{12}.$$

12. Let T be a bounded linear operator on a Hilbert space H, and assume that $\|T^n\| \leq 1$ for some natural number n. Prove the existence of an invertible linear operator A on H such that $\|ATA^{-1}\| \leq 1$.