Homework # 3, due Fri, Sep 24th.

1. Prove that there are no *odd* positive integers x, y, and z which satisfy the Pythagorean relation

$$(x+y)^2 + (x+z)^2 = (y+z)^2.$$

2. Each cell in a 5×41 rectangular grid is colored either red or blue. Prove that some 3 rows and 3 columns must intersect in 9 cells of the same color.

3. Find the sequence (a_n) if $a_0 = 1$ and

$$\sum_{k=0}^{n} a_k a_{n-k} = 1, \qquad n \geq 1.$$

4. Determine the last digit of

$$17^{17^{17}}$$

(in the decimal system).

5. Evaluate

$$\int_{-\infty}^{\infty} \frac{\mathrm{dx}}{1+x^{2n}}$$

where n is a positive integer.

6. Prove that the Taylor coefficients at the origin of the function

$$f(z) := \frac{z}{e^z - 1}$$

are rational numbers.

7. For which complex numbers z does the series

$$\sum_{n=0}^{\infty} \left(\frac{z^n}{n!} + \frac{n^2}{z^n} \right)$$

converge?

8. Let $A =: (a_{ij})_{i,j=1}^{n}$ be a real $n \times n$ matrix with nonnegative entries such that

$$\sum_{j=1}^{n} a_{ij} = 1, \qquad 1 \leq i \leq n.$$

Prove that no eigenvalue of A has an absolute value greater than 1.

9. Let $|U|$, $\sigma(U)$ and $\pi(U)$ denote the number of elements, the sum, and the product, respectively, of a finite set of positive integers. (If U is the empty set, then $|U| = 0$, $\sigma(U) = 0$, and $\pi(U) = 1$.) Let S be a finite set of positive integers. Prove that

$$\sum_{U \subseteq S} (-1)^{|U|} \binom{m - \sigma(U)}{|S|} = \pi(S)$$

for all integers $m \geq \sigma(S)$.

10. A square matrix is *nilpotent* if $A^k = 0$ for some positive integer k. Prove: if A and B are nilpotent matrices and $AB = BA$, then $A + B$ is nilpotent.

11. Let g be a continuous real valued function on $[0, 1]$. Prove that there exists a continuous real-valued function on $[0, 1]$ satisfying the equation

$$f(x) - \int_0^x f(x - t) e^{-t^2} \mathrm{dt} = g(x).$$

12.Suppose P is a polynomial of degree $3n$ such that

$$\begin{aligned} P(0) &= P(3) = \cdots = P(3n-3) = P(3n) = 2 \\ P(1) &= P(4) = \cdots = P(3n-2) = 1 \\ P(2) &= P(5) = \cdots = P(3n-1) = 0 \end{aligned}$$

If $P(3n + 1) = 730$, determine n.