Homework # 2, due Fri, Sep 17th.

1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

2. Each of the numbers x_1, x_2, …, x_n is either 1 or -1. If the sum

$$S := x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_6 + \cdots + x_nx_1x_2x_3 = 0,$$

prove that n must be a multiple of 4.

3. Prove that

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$$

for all positive integers $n \geq 2$.

4. Find all real-valued continuously differentiable functions on the real line such that, for all x,

$$(f(x))^2 = \int_0^x \left((f(t))^2 + (f'(t))^2\right) \mathrm{dt} + 2004.$$

5. For a partition π of $\{1,2,3,4,5,6,7,8,9\}$, let $\pi(x)$ be the number of elements in the part containing x. Prove that for any two partitions π and π', there are 2 distinct numbers x and y in $\{1,2,3,4,5,6,7,8,9\}$ such that $\pi(x) = \pi(y)$ and $\pi'(x) = \pi'(y)$.

6. Evaluate

$$\int_2^4 \frac{\sqrt{\ln(9-x)}\mathrm{dx}}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}.$$

7. Let a_0, …, a_n be arbitrary numbers and $A_k := \sum_{j\leq k} a_j$. Prove that

$$\begin{vmatrix} A_0 & A_0 & A_0 & A_0 & \cdots & A_0 \\ A_0 & A_1 & A_1 & A_1 & \cdots & A_1 \\ A_0 & A_1 & A_2 & A_2 & \cdots & A_2 \\ A_0 & A_1 & A_2 & A_3 & \cdots & A_3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ A_0 & A_1 & A_2 & A_3 & \cdots & A_n \end{vmatrix} = a_0a_1a_2\cdots a_n.$$

8. A composite (positive integer) is a product ab with a and b not necessarily distinct integers bigger than 1. Show that every composite is expressible as $xy + xz + yz + 1$ with x, y, z positive integers.

9. Let $(x_n)_{n\geq 0}$ be a sequence of nonzero real numbers such that

$$x_n^2 - x_{n-1}x_{n+1} = 1$$

for $n = 1, 2, 3, \ldots$. Prove that there exists a real number a such that

$$x_{n+1} = ax_n - x_{n-1}$$

for all $n \geq 1$.

10. Find all pairs of integers satisfying $0 < a < b$ and $a^b = b^a$.

11. Let k be a positive integer. For which values of the real number c does the differential equation

$$\frac{d^2x}{dt^2} - 2c\frac{dx}{dt} + x = 0$$

have a nontrivial solution satisfying $x(0) = x(2\pi k) = 0$?

12. For a permutation $a := (a_1, a_2, \ldots, a_n)$ of $\{1, 2, \ldots, n\}$, let

$$S(a) := (a_1 - a_2)^2 + (a_2 - a_3)^2 + \cdots + (a_{n-1} - a_n)^2.$$

what is the average value of S taken over all permutations?