1. A right circular cone has base of radius 1 and height 3. A cube is inscribed in the cone so that one face of the cube is contained in the base of the cone. What is the side-length of the cube?

2. Each of the numbers \(x_1, x_2, \ldots, x_n\) is either 1 or -1. If the sum
\[
S := x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_6 + \cdots + x_nx_1x_2x_3 = 0,
\]
prove that \(n\) must be a multiple of 4.

3. Prove that
\[
\sin \frac{\pi}{n} - \sin \frac{2\pi}{n} - \sin \frac{3\pi}{n} - \cdots - \sin \frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}
\]
for all positive integers \(n \geq 2\).

4. Find all real-valued continuously differentiable functions on the real line such that, for all \(x\),
\[
(f(x))^2 = \int_0^x (f(t))^2 + (f'(t))^2 \, dt + 2004.
\]

5. For a partition \(\pi\) of \(\{1, 2, 3, 4, 5, 6, 7, 8, 9\}\), let \(\pi(x)\) be the number of elements in the part containing \(x\). Prove that for any two partitions \(\pi\) and \(\pi'\), there are 2 distinct numbers \(x\) and \(y\) in \(\{1, 2, 3, 4, 5, 6, 7, 8, 9\}\) such that \(\pi(x) = \pi(y)\) and \(\pi'(x) = \pi'(y)\).

6. Evaluate
\[
\int_2^4 \frac{\sqrt{\ln(9-x)}}{\sqrt{\ln(9-x) + \ln(x+3)}} \, dx.
\]

7. Let \(a_0, \ldots, a_n\) be arbitrary numbers and \(A_k := \sum_{j \leq k} a_j\). Prove that
\[
\begin{bmatrix}
A_0 & A_0 & A_0 & \cdots & A_0 \\
A_0 & A_1 & A_1 & \cdots & A_1 \\
A_0 & A_1 & A_2 & \cdots & A_2 \\
A_0 & A_1 & A_2 & A_3 & \cdots & A_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_0 & A_1 & A_2 & A_3 & \cdots & A_n
\end{bmatrix} = a_0a_1a_2\cdots a_n.
\]

8. A composite (positive integer) is a product \(ab\) with \(a\) and \(b\) not necessarily distinct integers bigger than 1. Show that every composite is expressible as \(xy + xz + yz + 1\) with \(x, y, z\) positive integers.
9. Let \((x_n)_{n \geq 0}\) be a sequence of nonzero real numbers such that
\[x_n^2 - x_{n-1}x_{n+1} = 1\]
for \(n = 1, 2, 3, \ldots\). Prove that there exists a real number \(a\) such that
\[x_{n+1} = ax_n - x_{n-1}\]
for all \(n \geq 1\).

10. Find all pairs of integers satisfying \(0 < a < b\) and \(a^b = b^a\).

11. Let \(k\) be a positive integer. For which values of the real number \(c\) does the differential equation
\[\frac{d^2x}{dt^2} - 2c\frac{dx}{dt} + x = 0\]
have a nontrivial solution satisfying \(x(0) = x(2\pi k) = 0\)?

12. For a permutation \(a := (a_1, a_2, \ldots, a_n)\) of \(\{1, 2, \ldots, n\}\), let
\[S(a) := (a_1 - a_2)^2 + (a_2 - a_3)^2 + \cdots + (a_{n-1} - a_n)^2.\]
what is the average value of \(S\) taken over all permutations?