Homework # 1, due Fri, Sep 10th.

1. Prove that if f is a polynomial with integer coefficients, and there exists an integer k such that none of the integers $f(1)$, $f(2)$, …, $f(k)$ is divisible by k, then $f(x)$ has no integer root.

2. Find all continuous positive functions f defined on the interval $[0,1]$ such that

$$\int_0^1 f(x)\mathrm{dx} = 1$$
$$\int_0^1 f(x)x\mathrm{dx} = \alpha$$
$$\int_0^1 f(x)x^2\mathrm{dx} = \alpha^2$$

where α is a given real number.

3. Find, with explanation, the maximum value of the function $f(x) = x^3 - 3x$ on the set of all real numbers x satisfying the condition $x^4 + 36 \leq 13x^2$.

4. Prove that, for all positive integers n,

$$\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}.$$

5. Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2+1}, \qquad n = 1, 2, 3, \ldots,$$

compute the values of the derivatives $f^{(k)}(0)$, $k = 0, 1, 2, 3, \ldots$.

6. Evaluate the determinant

$$\begin{vmatrix} a_1^2 + x & a_1a_2 & a_1a_3 & \cdots & a_1a_n \\ a_2a_1 & a_2^2 + x & a_2a_3 & \cdots & a_2a_n \\ a_3a_1 & a_3a_2 & a_3^2 + x & \cdots & a_3a_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_na_1 & a_na_2 & a_na_3 & \cdots & a_n^2 + x \end{vmatrix}.$$

7. Delete from the harmonic series

$$\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$$

all terms whose denominators contain digit 9. Prove that the obtained series converges.

8. Let f be a real continuous function defined on the set of real numbers. Suppose that for every $\varepsilon \in [0, 1)$

$$\lim_{n \to \infty} f(\varepsilon + n) = 0 \qquad \text{(where } n \text{ runs over all positive integers).}$$

Prove or disprove that $\lim_{x \to \infty} f(x) = 0$.

9. Show that the improper integral

$$\lim_{B \to \infty} \int_0^B \sin(x) \sin(x^2) \mathrm{dx}$$

converges.

10. Four points are chosen at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points? (It is understood that each point is independently chosen relative to a uniform distribution on the sphere.)

11. Let p be an odd prime and let $\mathbb{Z}_p$ denote the field of integers modulo p. How many elements are in the set

$$\{x^2 : x \in \mathbb{Z}_p\} \cap \{y^2 + 1 : y \in \mathbb{Z}_p\}?$$

12. Let G be a finite set of real $n \times n$ matrices $\{M_j\}$, $j = 1, \ldots, r$, which form a group under matrix multiplication. Suppose that $\sum_{j=1}^{r} \mathrm{tr}(M_j) = 0$, where $\mathrm{tr}(A)$ denotes the trace of the matrix A. Prove that $\sum_{j=1}^{r} M_j$ is the $n \times n$ zero matrix.