Hints for homework # 6.

1. Straightforward.
2. Determine the distance between the least and the biggest square. This restricts your choices quite a bit.
3. Pigeonhole principle.
4. Use vectors.
5. Use congruences mod 3 and mod 5.
6. Use Binomial theorem.
7. The limit exists and is directly related to the sum of the doubly infinite series \( \sum_{n=1}^{\infty} a_n \).
8. Examine the behavior of the sequence \( x, x^2 + c, (x^2 + c)^2 + c, \ldots \), for various values of \( x \) and \( c \).
9. Begin by considering two subsets, \( A \) and \( B \) and by putting into \( A \) all the terms that do not divide any other term of \( S \).
10. 
11. Hard. First solve the following auxiliary problem: Let \( a \) and \( b \) be integers and let \( f \) be a function that is positive in the interval \( a \leq x \leq b \). Find the number of integer points in the region

\[
\begin{align*}
a \leq x \leq b, \quad 0 < y \leq f(x).
\end{align*}
\]

12. May be hard. Consider a new inner product defined by

\[
\langle x, y \rangle_{\text{new}} := \sum_{j=0}^{n-1} \langle T^j x, T^j y \rangle.
\]

The new space is isomorphic to the original space.