Hints for homework # 4.

1. The remainder is a polynomial of degree 1.
2. Easy.
3. What recurrence relation does the exponential sequence from the problem satisfy?
4. Easy.
5. These symmetric functions are related to some polynomial.
6. Connect the series to a certain integral.
7. High level; hard. For $a \in (0,1)$, consider the integral average

   $$g_a(z) := \frac{1}{4a^2} \int_{-a}^{a} \int_{-a}^{a} f(z + x + \mathrm{i}y) \mathrm{d}x\mathrm{d}y$$

   where $\mathbb{R}^2$ is identified with $\mathbb{C}$. Its properties place it in a certain set of uniformly bounded and uniformly equicontinuous functions. Then apply Ascoli-Arzela to derive that g_a is constant.
8. What is a preimage of a set where a function takes on its local maximum or minimum?
9. Use generating functions.
10. Rather involved. First count the number of 5-term arithmetic progressions in which the difference d is fixed. Next, find possible values of d. Then count the number of all possible 5-term arithmetic progressions. Finally, find the number of ways those 5 numbers can be arranged so that the first 3 form an arithmetic progression too. Then prove the lower bound on the obtained probability.
11. Hard. First solve the following auxiliary problem: Let a and b be integers and let f be a function that is positive in the interval $a \leq x \leq b$. Find the number of integer points in the region

    $$a \leq x \leq b, \qquad 0 < y \leq f(x).$$
12. May be hard. Consider a new inner product defined by

    $$\langle x, y \rangle_{\text{new}} := \sum_{j=0}^{n-1} \langle T^j x, T^j y \rangle.$$

    The new space is isomorphic to the original space.