Polynomial equations and symmetric functions.

While algorithms for solving polynomial equations of degree at most 4 exist, there are in general no such algorithms for polynomials of higher degree. A polynomial equation to be solved at an Olympiad is usually solvable by using the Rational Root Theorem (see the earlier handout RATIONAL AND IRRATIONAL NUMBERS), symmetry, special forms, and/or symmetric functions.

Here are, for the record, algorithms for solving 3rd and 4th degree equations.

Algorithm for solving cubic equations. The general cubic equation

$$c_0 + c_1x + c_2x^2 + c_3x^3 = 0$$

can be transformed (by dividing by c_3 and letting $z := x + \frac{c_2}{3c_3}$) into an equation of the form

$$z^3 + pz + q = 0.$$

To solve this equation, we substitute $x = u + v$ to obtain

$$u^3 + v^3 + (u + v)(3uv + p) + q = 0.$$

Note that we are free to restrict u and v so that $uv = -p/3$. Then u^3 and v^3 are the roots of the equation $z^2 + qz - p^3/27 = 0$. Solving this equation, we obtain

$$u^3,\ v^3 = -\frac{q}{2} \pm \sqrt{2},$$

where

$$R := \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3.$$

Now we may choose cube roots so that

$$A := \sqrt[3]{-\frac{q}{2} + \sqrt{R}}, \qquad B := \sqrt[3]{-\frac{q}{2} - \sqrt{R}}.$$

Then $A + B$ is a solution. It is easily checked that the other pairs are obtained by rotating A and B in the complex plane by angles $\pm 2\pi/3$, $\mp 2\pi/3$. So, the full set of solutions is

$$\{A + B,\ \omega A + \overline{\omega} B,\ \overline{\omega} A + \omega B\} \qquad \text{where } \omega := e^{2\pi i/3}.$$

Ferrari's method of solving quartic equations. The general quartic equation is reduced to a cubic equation called the *resolvent.* write the quartic equation as

$$x^4 + 2ax^3 + b^2 + 2cx + d = 0.$$

Transpose to obtain

$$x^4 + 2ax^3 = -bx^2 - 2cx - d$$

and then adding $2rx^2+(ax+r)^2$ to both sides makes the left-hand side equal to $(x^2+ax+r)^2$. If r can be chosen to make the right-hand side a perfect square, then it will be easy to find all solutions. The right-hand side,

$$(2r + a^2 - b)x^2 + 2(ar - c)x + (r^2 - d),$$

is a perfect square if and only if its discriminant is zero. Thus we require

$$2r^3 - br^2 + 2(ac - d)r + (bd - a^2 d - c^2) = 0.$$

This is the cubic resolvent.

Reciprocal or palindromic equations. If the equation the form $a_0+a_1x+\cdots+a_nx^n = 0$ and $a_j = a_{n-j}$ for all $j = 0, \ldots, n$, it is called *palindromic*. For even n, the transformation $z := x + \frac{1}{x}$ reduces the equation to a new one of degree $n/2$. After finding all solutions z_j, the solutions of the original equation are found by solving quadratic equations $x + \frac{1}{x} = z_j$.

Examples.

1. Solve $x^4 + 2x^3 + 7x^2 + 6x + 8 = 0$.
2. Solve $x^4 + 2ax^3 + bx^2 + 2ax + 1 = 0$.
3. Solve $x^3 - 3x + 1 = 0$.
4. Solve $x^4 - 26x^2 + 72x - 11 = 0$.
5. Solve $z^4 - 2z^3 + z^2 - a = 0$ and find values of a for which all roots are real.

Definitions. A function of n variables is *symmetric* if it is invariant under any permutation of its variables. The kth *elementary symmeric function* is defined by

$$\sigma_k(x_1, \ldots, x_n) := \sum x_{i_1} x_{i_2} \cdots x_{i_k},$$

where the sum is taken over all $\binom{n}{k}$ choices of the indices i_1, i_2, …, i_k from the set $\{1, 2, \ldots, n\}$.

Symmetric function theorem. Every symmetric polynomial function of x_1, …, x_n is a polynomial function of σ_1, …, σ_n. The same conclusion holds with "polynomial" replaced by "rational function".

Theorem. Let x_1, …, x_n be the roots of the polynomial equation

$$x^n + c_1 x^{n-1} + \cdots + c_n = 0,$$

and let σ_k be the kth elementary symmetric function of x_1, …, x_n. Then

$$\sigma_k = (-1)^k c_k, \qquad k = 1, \ldots, n.$$

Newton's formula for power sums. Let

$$S_p := x_1^p + x_2^p + \cdots + x_n^p, \qquad p \in \mathbb{N},$$

where $x_1, \ldots, x_n$ are the roots of

$$x^n + c_1 x^{n-1} + \cdots + c_n = 0.$$

Then

$$\begin{aligned}
S_1 + c_1 &= 0 \\
S_2 + c_1 S_1 + 2c_2 &= 0 \\
S_3 + c_1 S_2 + c_2 S_1 + 3c_3 &= 0 \\
\cdots\cdots\cdots & \\
S_n + c_1 S_{n-1} + \cdots + c_{n-1} S_1 + n c_n &= 0 \\
S_p + c_1 S_{p-1} + \cdots + c_n S_{p-n} &= 0, \quad p > n.
\end{aligned}$$

Examples.

1. Find all solutions of the system

$$\begin{aligned}
x + y + z &= 0 \\
x^2 + y^2 + z^2 &= 6ab \\
x^3 + y^3 + z^3 &= 3(a^3 + b^3).
\end{aligned}$$

2. If

$$\begin{aligned}
x + y + z &= 1 \\
x^2 + y^2 + z^2 &= 2 \\
x^3 + y^3 + z^3 &= 3,
\end{aligned}$$

   determine the value of $x^4 + y^4 + z^4$.

3. Let $G_n := a^n \sin(nA) + b^n \sin(nB) + c^n \sin(nC)$, where a, b, c, A, B, C are real numbers and $A + B + C$ is a multiple of π. Prove that if $G_1 = G_2 = 0$, then $G_k = 0$ for all $k \in \mathbb{N}$.

4. Find a cubic equation whose roots are the cubes of the roots of $x^3 + ax^2 + bx + c = 0$.

5. Find all values of the parameter a such that all roots of the equation

$$x^6 + 3x^5 + (6 - a)x^4 + (7 - 2a)x^3 + (6 - a)x^2 + 3x + 1 = 0$$

   are real.

6. A student awoke at the end of an algebra class just in time to hear the teacher say, "...and I give you a hint that the roots form an arithmetic progression." Looking at the board, the student discovered a fifth degree equation to be solved for homework, but he had time to copy only

$$x^5 - 5x^4 - 35x^3 +$$

   before the teacher erased the blackboard. He was able to find all roots anyway. What are the roots?