Diophantine equations.

Some solution methods:

- factoring;
- congruence;
- using discriminant for quadratic equations;
- Fermat's method of infinite descent;
- special forms.

Pythagorean equation.

$$x^2 + y^2 = z^2$$

A *primitive solution* of this equation is one for which x, y and z have no factor in common.

Theorem. All positive primitive solutions of the Phythagorean equation where y is even are given by $x = a^2 - b^2$, $y = 2ab$, $z = a^2 + b^2$, where a and b are of opposite parity, $(a, b) = 1$, and $a > b > 0$.

Theorem. All rational points on the circle

$$x^2 + y^2 = 1$$

are given by the formula

$$x = \frac{1 - t^2}{1 + t^2}, \qquad y = \frac{2t}{1 + t^2}$$

where t runs over all rationals.

Examples.

1. Prove that the equation $x^2 = 3y^2 + 8$ has no solutions in integers x, y.
2. Find all solutions of the equations $a^2 + b^2 + c^2 = a^2b^2$ in natural numbers a, b, c.
3. Let a and b be positive integers such that $(1 + ab)|(a^2 + b^2)$. Show that the integer $(a^2 + b^2)/(1 + ab)$ must be a perfect square.
4. Prove that the equation $x^4 + y^4 = z^2$ has no solution in natural numbers.
5. Prove that $n^2 + (n + 1)^2$ is a perfect square for infinitely many natural numbers n.
6. Let $p > 5$ be prime. Prove that the equation $x^4 + 4^x = p$ has no integer solutions.
7. Find all solutions of $1! + 2! + \cdots + x! = y^2$ in integers x, y.