Rolle's theorem and Descartes' rule of signs.

Here we consider real functions of a real variable x; in particular, the coefficients of polynomials and power series are all assumed to be real. Moreover, unless stipulated otherwise, all functions under consideration are assumed to be analytic in given intervals. All roots are counted according to their multiplicity.

Definition. Given a finite or infinite sequence a_0, a_1, a_2, …, we call index $m \geq 1$ the location of a *sign change* if

$$a_{m-1}a_m < 0$$

or

$$a_{m-1} = \cdots = a_{m-k+1} = 0, \qquad a_{m-k}a_m < 0, \qquad m \geq k \geq 2.$$

The elements a_{m-1} and a_m in the former case and a_{m-k} and a_m in the latter case *form a sign change.* The number of sign changes of a sequence does not change if its zero elements are omitted.

Counting sign changes.

1. The number of sign changes does not increase if some elements of the sequence are deleted.

2. The sequence a_0, $a_0 + a_1$, $a_1 + a_2$, …, a_n has no more sign changes than the sequence a_0, a_1, …, a_n.

3. If an infinite sequence a_0, …, a_n, … has a finite number W of sign changes, then the sequence

$$a_0,\ a_0 + a_1,\ a_0 + 2a_1 + a_2,\ \ldots,\ a_0 + \binom{n}{1}a_1 + \binom{n}{2}a_2 + \cdots + a_n,\ \ldots$$

has at most W sign changes.

4. Suppose the values $f(a)$ and $f(b)$ are both nonzero. Then the interval $a < x < b$ contains an even or odd number of zeroes of f depending on whether $f(a)$ and $f(b)$ have the sam or opposite signs.

5. Suppose a_j and a_k are nonzero. The part a_j, a_{j+1}, …, a_k has an even or odd number of sign changes depending on whether a_j and a_k have the same or opposite signs.

6. **[Rolle's theorem.]** Let a and b be consecutive zeroes of f, i.e., $f(a) = f(b) = 0$ and $f(x) \neq 0$ for $a < x < b$. Then the derivative f' has an odd number of zeroes (in particular, at least one).

7. If $j + 1$ and $k + 1$ are consecutive locations of sign changes, then the difference sequence

$$a_{j+1} - a_j,\ a_{j+2} - a_{j+1},\ \ldots,\ a_k - a_{k-1},\ a_{k+1} - a_k$$

has an odd number of sign changes (in particular, at least one).

8. If f has N zeroes in the interval a, b, then f' has at least $N - 1$ zeroes there. The theorem holds for closed, open, or half-open intervals; the intervals can even consist of one point.

9. If the sequence a_0, a_1, …, a_n has W sign changes, then the difference sequence

$$a_1 - a_0,\ a_2 - a_1,\ \ldots,\ a_n - a_{n-1}$$

has at least $W - 1$ sign changes.

10. If f has N intervals in a finite interval $a < x < b$ and satisfies one of the conditions

$$\text{sgn} f(a) = \text{sgn} f'(a) \neq 0, \qquad \text{sgn} f(b) = -\text{sgn} f'(b) \neq 0,$$

then f' has at least N zeroes in the same interval. If both conditions are met, then $'$ has at least $N + 1$ zeroes in a, b.

11. If the finite sequence a_0, a_1, a_2, …, a_n has W sign changes, then the sequence

$$a_0,\ a_1 - a_0,\ a_2 - a_1,\ \ldots,\ a_n - a_{n-1}, -a_n$$

has at least $W + 1$ sign changes (except for the trivial case when the original sequence consists of zeroes).

12. If $\lim_{x\to+\infty} f(x) = 0$, then f' has at least as many roots as f between a and $+\infty$. (The same holds for $-\infty$.)

13. If $\lim a_n = 0$, then the sequence

$$a_0,\ a_1 - a_2,\ a_2 - a_1,\ \ldots,\ a_n - a_{n-1}, \ldots$$

has more sign changes than the original sequence a_0, a_1, ….

14. Suppose f has N zeroes in the interval $0 < x < \infty$. Then the function

$$\alpha f + f',$$

where α is real, has at least $N - 1$ zeroes in the same interval. Moreover, if $\lim_{x\to+\infty} e^{\alpha x} f(x) = 0$, then the function has at least N zeroes.

15. Suppose the infinite sequence a_0, a_1, … has W sign changes. Then the sequence

$$\alpha a_0,\ \alpha a_1 - a_0,\ \alpha a_2 - a_1,\ \ldots,\ \alpha a_n - a_{n-1},\ \ldots$$

where $\alpha > 0$, has at least W sign changes. Moreover, if $\lim_{n\to\infty} a_n \alpha^n = 0$, then the obtained sequence has at least $W + 1$ sign changes.

16. If the function f has N zeroes in the interval $0 < x < \infty$, then the function $\int_0^x f(t)\mathrm{d}t$ has at most N zeroes in the same interval.

Derivation of Descartes' rule.

17. If the function a_0, a_1, …, a_n, … has W sign changes, then the sequence

$$a_0,\ a_0 + a_1,\ a_0 + a_1 + a_2,\ \ldots,\ a_0 + a_1 + \cdots + a_n,\ \ldots$$

has at most W sign changes.

18. Let $\alpha > 0$. The transition from the polynomial

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

to the polynomial

$$(\alpha - x)(a_0 + a_1 x + \cdots a_n x^n) = \alpha a_0 + (\alpha a_1 - a_0)x + (\alpha a_2 - a_1)x^2 + \cdots - a_n x^{n+1}$$

increases the number of sign changes in the coefficient sequence by an odd number.

19. Let $\alpha > 0$. The transition from the power series

$$a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots$$

to the series

$$(\alpha - x)(a_0 + a_1 x + \cdots a_n x^n) = \alpha a_0 + (\alpha a_1 - a_0)x + (\alpha a_2 - a_1)x^2 + \cdots$$

does not decrease the number of sign changes. Moreover, the number increases if the first series converges for $x = \alpha$.

20. Let $\alpha > 0$. The transition from the power series

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

to the series

$$(\alpha + x)(a_0 + a_1x + \cdots a_nx^n) = \alpha a_0 + (\alpha a_1 + a_0)x + (\alpha a_2 + a_1)x^2 + \cdots$$

does not increase the number of sign changes.

21. The transition from the power series

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots$$

to the series

$$\frac{1}{1-x}(a_0 + a_1x + \cdots a_nx^n) = a_0 + (a_0 + a_1)x + (a_0 + a_1 + a_2)x^2 + \cdots$$

does not increase he number of sign changes.

22. **Theorem [Descartes' rule of signs].** Let N be the number of positive zeroes of a polynomial $a_0 + a_1x + \cdots + a_nx^n$ and let W be the number of sign changes in the sequence of its coefficients. Then $W - N$ is an even nonnegative number.

23. **Theorem [Descartes' rule of signs for analytic functions].** Let ϱ be the radius of convergence of the series $a_0 + a_1x + \cdots + \cdots + a_nx^n + \cdots$, let N be the number of its zeroes on the interval $0 < x < \varrho$ and let W be the number of sign changes in its sequence of coefficients. Then $N \leq W$.

23. Show that the series

$$2 - \frac{x}{1 \cdot 2} - \frac{x^2}{2 \cdot 3} - \frac{x^3}{3 \cdot 4} - \cdots$$

does not have zeroes in its disk of convergence. (So the result that $N - W$ is even does not generalize automatically to power series.)

24. [Continuation of 23.] If $\varrho = \infty$ or if ϱ is finite but the series $a_0 + a_1\varrho + a_2\varrho^2 + \cdots$ diverges, then the difference $W - N$ is a nonnegative even number.

Some applications.

25. Let $\lambda \in (0, 1)$. Then the equation

$$1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} = \lambda e^x$$

has only one positive root.

26. The function $x^{-5}(e^{1/x} - 1)^{-1}$ tends to zero when $x \to 0$ and $x \to \infty$ and has exactly one maximum and no minima in between.

27. Let the radius of convergence of the series $a_0 + a_1x + a_2x^2 + \cdots$ be greater than or equal to 1. Show that the number of zeroes of that power series in the interval $0 < x < 1$ does not exceed the number of sign changes in the sequence

$$a_0,\ a_0 + a_1,\ a_0 + a_1 + a_2,\ \ldots$$

28. The equation

$$\left(\frac{1}{19}\right)x + \left(\frac{2}{19}\right)x^2 + \cdots + \left(\frac{18}{19}\right)x^{18} = 0$$

where $\left(\frac{n}{p}\right)$ is as usual the Legendre symbol, has exactly one positive root. (Notice the symmetry of coefficients, use 27, 21.)

29. Suppose a_0, $a_n \neq 0$ and $2m$ consecutive coefficients of the polynomial $a_0 + a_1x + \cdots + a_nx^n$ are zero. Then the polynomial has at least $2m$ imaginary roots.

30. Let ν_1, ν_2, ..., ν_n be integers such that $0 \leq \nu_1 < \nu_2 < \cdots < \nu_n$. Let $0 < \alpha_1 < \alpha_2 < \cdots < \alpha_n$. Show that

$$\begin{vmatrix} \alpha_1^{\nu_1} & \alpha_1^{\nu_2} & \cdots & \alpha_1^{\nu_n} \\ \alpha_2^{\nu_1} & \alpha_2^{\nu_2} & \cdots & \alpha_2^{\nu_n} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n^{\nu_1} & \alpha_n^{\nu_2} & \cdots & \alpha_n^{\nu_n} \end{vmatrix} > 0.$$