Solutions to MATH 54 2nd mock midterm test

1. Subtracting the first equation from the second leads to $-4x_2 = 0$, while subtracting the first equation from the third leaves $x_2 = 1$, so the system is obviously inconsistent.

The least-squares approximation x^* to $Ax = b$ satisfies $A^T A x^* = A^T b$; in this case $\operatorname{rank} A^T A = \operatorname{rank} A = 2$, so the solution is unique:

$$\underbrace{\begin{bmatrix} 3 & 0 \\ 0 & 14 \end{bmatrix}}_{A^T A} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{A^T (0\ 0\ 1)^T} \implies x^* = \begin{bmatrix} 1/3 \\ 1/7 \end{bmatrix}$$

2. The condition $AB = 0$ implies $ABx = 0$ for any vector x. In other words, the matrix A maps every vector in the column space of B to the zero vector. Hence the column space of B is contained in the null space of A.

Now, since $\operatorname{Col} B \subseteq \operatorname{Nul} A$, we have $\operatorname{rank} B = \dim \operatorname{Col} B \leq \dim \operatorname{Nul} A$. On the other hand, the Rank-Nullity Theorem says

$$\operatorname{rank} A + \dim \operatorname{Nul} A = n.$$

But then

$$\operatorname{rank} A + \operatorname{rank} B \leq \operatorname{rank} A + \dim \operatorname{Nul} A = n.$$

3. In order to check that we have a real inner product space, we need to verify that the bracket is linear (say in the first variable), symmetric, and positive. Linearity follows from linearity of the definite integral in its integrand, symmetry follows from the fact that f and g can be interchanged in the product, and positive definiteness follows from the fact that if f is a non-zero continuous function, then it will be bounded away from zero on some interval $I \subseteq [0, 2\pi]$, and the same will be true for the non-negative function $f(t)^2$, so

$$\langle f, f \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(t)^2\, dt \geq \frac{1}{2\pi} \int_I f(t)^2\, dt > 0.$$

If $W = \operatorname{span}\{1, \sin t, \cos t\}$, one can verify that the given functions 1, $\sin t$ and $\cos t$ already form an orthogonal basis of W, but to get an orthonormal basis they need to be normalized. The following table of indefinite integrals will be useful:

$$\begin{aligned}
\textstyle\int \sin t\, dt &= -\cos t + C & \int \cos t\, dt &= \sin t + C \\
\textstyle\int \sin^2 t\, dt &= \tfrac{1}{2}t - \tfrac{1}{4}\sin 2t + C & \int \cos^2 t\, dt &= \frac{1}{2}t + \frac{1}{4}\sin 2t + C \\
\textstyle\int t\sin t\, dt &= -t\cos t + \sin t + C & \int t\cos t\, dt &= t\sin t + \cos t + C.
\end{aligned}$$

We have

$$\int_0^{2\pi} dt = 2\pi, \qquad \frac{1}{2\pi}\int_0^{2\pi} (\sin t)^2\, dt = \frac{1}{2\pi}\int_0^{2\pi} (\cos t)^2\, dt = \frac{1}{2},$$

so an orthonormal basis of W is given by $\{1, \sqrt{2}\sin t, \sqrt{2}\cos t\}$. Therefore the orthogonal projection of t onto W equals

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi} t\, dt + \left(\frac{1}{2\pi}\int_0^{2\pi} t(\sqrt{2}\sin t)\, dt\right)(\sqrt{2}\sin t) + \\
&\qquad + \left(\frac{1}{2\pi}\int_0^{2\pi} t(\sqrt{2}\cos t)\, dt\right)(\sqrt{2}\cos t) = \pi - 2\sin t.
\end{aligned}$$

4. We calculate

$$T(1) = (t^2)' = 2t \qquad T(t) = (t^3)' = 3t^2 \qquad T(t^2) = (t^4)' = 4t^3.$$

The matrix representation of T is therefore just

$$[T] = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

(into each column we put the coordinates of the image of the corresponding basis vector). We see that $\operatorname{Ker} T = \{0\}$, hence the empty set forms a basis of $\operatorname{Ker} T$.

5. The characteristic polynomial of A is $\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1)$. An eigenvector for $\lambda = -1$ is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and an eigenvector for $\lambda = 2$ is $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Putting these into the columns of a matrix

$$P = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix},$$

we have

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$$

and

$$P^{-1}AP = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

So, if D denotes this diagonal matrix, then

$$A^{1000} = PD^{1000}P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} (-1)^{1000} & 0 \\ 0 & 2^{1000} \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2^{1001} + 1 & 2^{1001} - 2 \\ 2^{1000} - 1 & 2^{1000} + 2 \end{bmatrix}.$$