

Solutions to MATH 54 mock final test

1. Determine for which values of  $b_1, b_2, b_3$  the system

$$\begin{aligned} 2x_1 - 4x_2 - 2x_3 &= b_1 \\ -5x_1 + x_2 + x_3 &= b_2 \\ 7x_1 - 5x_2 - 3x_3 &= b_3 \end{aligned}$$

has a solution.

**Solution.** Perform row reduction on the augmented matrix of the system. This yields

$$\begin{aligned} \left[ \begin{array}{ccc|c} 2 & -4 & -2 & b_1 \\ -5 & 1 & 1 & b_2 \\ 7 & -5 & -3 & b_3 \end{array} \right] &\sim \left[ \begin{array}{ccc|c} 1 & -2 & -1 & \frac{b_1}{2} \\ -5 & 1 & 1 & b_2 \\ 7 & -5 & -3 & b_3 \end{array} \right] \sim \left[ \begin{array}{ccc|c} 1 & -2 & -1 & \frac{b_1}{2} \\ 0 & -9 & -4 & b_2 + \frac{5b_1}{2} \\ 0 & 9 & 4 & b_3 - \frac{7b_1}{2} \end{array} \right] \\ &\sim \left[ \begin{array}{ccc|c} 1 & -2 & -1 & \frac{b_1}{2} \\ 0 & -9 & -4 & b_2 + \frac{5b_1}{2} \\ 0 & 0 & 0 & b_3 + b_2 - b_1 \end{array} \right]. \end{aligned}$$

This shows that the row reduced system has a solution if and only if the last entry of the modified right-hand side,  $b_3 + b_2 - b_1$ , is zero. Since the row reduced system is equivalent to the original one, the same is true of the original system. **Answer:** whenever  $b_3 = b_1 - b_2$ .

2. Let  $T : \mathbb{P}_2 \rightarrow \mathbb{R}^3$  be the linear map taking a polynomial of degree at most 2 to its values at the points  $-1, 0$  and  $1$ :

$$T(p) := \begin{bmatrix} p(-1) \\ p(0) \\ p(1) \end{bmatrix}.$$

(a) Write the matrix of  $T$  in the standard bases.

(b) Using the result of (a), find a polynomial  $f \in \mathbb{P}_2$  such that  $f(-1) = 1, f(0) = 0, f(1) = 2$ .

**Solution.** (a) We need to compute the images of the standard basis vectors, i.e., the monomials  $1, x, x^2$ , under the map  $T$ . Since

$$T(1) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad T(x) = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad T(x^2) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

the matrix  $[T]$  of the linear map  $T$  with respect to the standard monomial basis in  $\mathbb{P}^2$  and the standard basis in  $\mathbb{R}^3$  is

$$[T] = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$

(b) To find a polynomial  $f(x) = a_0 + a_1x + a_2x^2$  with  $T(-1) = 1$ ,  $T(0) = 0$ ,  $T(1) = 2$ , we must solve the linear system  $[T]\vec{a} = \vec{b}$  where  $[T]$  is the matrix of the map  $T$  from part (a),

$$\vec{a} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \quad \text{and} \quad \vec{b} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} :$$

$$\begin{aligned} a_0 - a_1 - a_2 &= 1 \\ a_1 &= 0 \\ a_0 + a_1 + a_2 &= 2. \end{aligned}$$

Subtracting the third equation from the first, we determine that  $2a_2 = 1$ , hence  $a_2 = 1/2$ . The second equation gives  $a_1 = 0$ , and the first equation therefore gives  $a_3 = 3/2$ . Hence the polynomial whose values at  $-1, 0, 1$  are as  $1, 0, 2$  is  $\frac{1}{2}x + \frac{3}{2}x^2$ . **Answer:**  $f(x) = \frac{1}{2}x + \frac{3}{2}x^2$ .

3. Let  $S$  be the vector space of functions of the form

$$S := \{p(x)e^{2x} : p \in \mathbb{P}_2\}$$

and  $L$  be the differential operator  $L := (D - 2I)^2 = (D - 2I)(D - 2I)$  where  $I$  is the identity map and  $D = \frac{d}{dx}$  is the operator of differentiation. Is there a basis for  $S$  consisting of eigenvectors of  $L$ ?

**Solution.** The (standard) basis for  $S$  is  $\mathcal{B} = \{e^{2x}, xe^{2x}, x^2e^{2x}\}$ . We need to find out how  $L$  acts on each of the basis vectors (functions). We get

$$\begin{aligned} \left(\frac{d}{dx} - 2I\right)e^{2x} &= 0, \quad \text{hence} \quad L(e^{2x}) = 0 \\ \left(\frac{d}{dx} - 2I\right)(xe^{2x}) &= e^{2x}, \quad \text{hence} \quad L(xe^{2x}) = \left(\frac{d}{dx} - 2I\right)e^{2x} = 0, \\ \left(\frac{d}{dx} - 2I\right)(x^2e^{2x}) &= 2xe^{2x}, \quad \text{hence} \quad L(x^2e^{2x}) = \left(\frac{d}{dx} - 2I\right)(2xe^{2x}) = 2e^{2x}. \end{aligned}$$

Thus the matrix  $[L]$  of the operator  $L$  in the basis  $\mathcal{B}$  is

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

This matrix has all eigenvalues zero. If it were diagonalizable, it would be similar to the zero matrix of order 3. But the only matrix similar to the zero matrix is itself. Hence  $[L]$  is not diagonalizable, and  $S$  does not have a basis consisting of eigenvectors of  $L$ . **Answer:** No.

4. Using variation of parameters, find a particular solution to the equation

$$y'' + y = \sec t.$$

**Solution.** The general solution to the homogeneous version of the equation has the form  $c_1 \sin t + c_2 \cos t$ . To use variation of parameters, we replace the constants  $c_1, c_2$  by function  $c_1(t), c_2(t)$ . Differentiating once, we get

$$y'(t) = c_1(t) \cos t - c_2(t) \sin t + (c_1'(t) \sin t + c_2'(t) \cos t)$$

and set the part in parentheses to zero. With that setting, the second differentiation gives

$$y''(t) = -c_1(t) \sin t - c_2(t) \cos t + (c_1'(t) \cos t - c_2'(t) \sin t).$$

To satisfy out inhomogeneous equation, we must set the second expression in parentheses to  $\sec t$ . This gives the linear system

$$\begin{aligned} c_1'(t) \sin t + c_2'(t) \cos t &= 0 \\ c_1'(t) \cos t - c_2'(t) \sin t &= \sec t \end{aligned}$$

in the unknowns  $c_1'(t), c_2'(t)$ . Solving this system, we get  $c_1'(t) = 1, c_2'(t) = -\sin t \sec t$ . So  $c_1$  should be taken as any antiderivative of 1; take  $c_1(t) = t$ , and

$$c_2(t) = - \int \frac{\sin t dt}{\cos t} = \int \frac{d(\cos t)}{\cos t} = \ln(\cos t).$$

(Of course,  $c_2(t)$  is also defined up to a constant, but one function is good enough to form a particular solution.) **Answer:**  $y(t) = t \sin t + \ln(\cos t) \cdot \cos t$ .

5. Find a general solution to the equation

$$y^{(4)} + 4y = x^5 + 2x^4 - x^3 + 1.$$

**Solution.** The characteristic equation corresponding to this ODE is  $\lambda^4 + 4 = 0$ . It splits as  $(\lambda^2 + 2i)(\lambda^2 - 2i) = 0$  or, alternatively, as  $(\lambda^2 - 2\lambda + 2)(\lambda^2 + 2\lambda + 2) = 0$ . Either way, the four roots are  $\pm 1 \pm i$ . So a general solution to the homogeneous equation has the form

$$c_1 e^x \cos x + c_2 e^x \sin x + c_3 e^{-x} \cos x + c_4 e^{-x} \sin x.$$

Note that 0 is not a root of the characteristic equation. Therefore a particular solution to the inhomogeneous equation with the given right-hand side is of the form

$$y_p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5.$$

Differentiating this function four times, we get  $y_p^{(4)}(x) = 24a_4 + 120a_5 x$ . So, the coefficients of  $y_p^{(4)}(x) + y_p(x)$  must match those of  $x^5 + 2x^4 - x^3 + 1$ . This gives:  $a_5 = 1, a_4 = 2, a_3 = -1, a_2 = 0, a_1 = -120, a_0 = -47$ . Combining the particular solution to the inhomogeneous equation that we just found with a general solution to the homogeneous equation, we obtain, by the superposition principle, a general solution to the inhomogeneous equation. **Answer:**  $c_1 e^x \cos x + c_2 e^x \sin x + c_3 e^{-x} \cos x + c_4 e^{-x} \sin x + \frac{1}{4}x^5 + \frac{1}{2}x^4 - \frac{1}{4}x^3 - \frac{15}{2}x - \frac{11}{4}$ .

6. Do the functions  $\{e^{3x}, e^{-x}, e^{-4x}\}$  form a fundamental solution set for the differential equation

$$y''' + 2y'' - 11y' - 12y = 0?$$

**Solution.** The characteristic equation of the homogeneous ODE is  $\lambda^3 + 2\lambda^2 - 11\lambda - 12 = 0$ , which factors as  $(\lambda + 1)(\lambda - 3)(\lambda + 4) = 0$ . Hence the exponentials  $e^{-x}$ ,  $e^{3x}$ ,  $e^{-4x}$  indeed form a fundamental solution set for this equation.

**Alternative solution (sketch).** One can instead verify that each of the functions  $e^{3x}$ ,  $e^{-x}$ ,  $e^{-4x}$  solves the ODE and then compute the Wronskian of this system

$$\begin{vmatrix} e^{3x} & e^{-x} & e^{-4x} \\ 3e^{3x} & -e^{-x} & -4e^{-4x} \\ 9e^{3x} & e^{-x} & 16e^{-4x} \end{vmatrix} = e^{-2x} \begin{vmatrix} 1 & 1 & 1 \\ 3 & -1 & -4 \\ 9 & 1 & 16 \end{vmatrix} = -84e^{-2x}$$

and see that the Wronskian never vanishes. Hence, again, this is a fundamental solution set.

7. Consider the following system of second order differential equations in two functions  $y = y(t)$  and  $z = z(t)$ :

$$\begin{aligned} y'' + 16z &= 0 \\ z'' - y &= 0 \end{aligned}$$

- Convert the system to a system of *first order* differential equations in four functions  $x_1, x_2, x_3$ , and  $x_4$ .
- Use part (a) to produce four linearly independent solutions to the original system.

**Solution.** Denote  $x_1 := y$ ,  $x_2 := y'$ ,  $x_3 := z$ ,  $x_4 := z'$ . The the original system turns into the following first-order system:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -16 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}. \quad (1)$$

The characteristic equation of this matrix ODE is

$$0 = \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & -16 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} = -\lambda \begin{vmatrix} -\lambda & -16 & 0 \\ 0 & -\lambda & 1 \\ 0 & 0 & -\lambda \end{vmatrix} - \begin{vmatrix} 0 & -16 & 0 \\ 0 & -\lambda & 1 \\ 1 & 0 & -\lambda \end{vmatrix} = \lambda^4 + 16,$$

whose roots are  $\sqrt{2}(\pm 1 \pm i)$ . These are four distinct eigenvalues, hence there exist four nonzero constant vectors  $\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4$  (eigenvectors of the matrix in the right-hand side of (1)) such that the vector-valued functions  $\vec{v}_1 e^{\sqrt{2}(1+i)t}$ ,  $\vec{v}_2 e^{\sqrt{2}(1-i)t}$ ,  $\vec{v}_3 e^{\sqrt{2}(-1+i)t}$ ,  $\vec{v}_4 e^{\sqrt{2}(-1-i)t}$  form a fundamental solution set to the system (1).

As we are interested in the resulting formulæ for the original functions  $y$  and  $z$ , we can perform the following shortcut: instead of finding the eigenvectors and the resulting  $x_1$  through  $x_4$ , we can directly substitute solutions of the form  $y = ce^{\sqrt{2}(\pm 1 \pm i)t}$ ,  $z = de^{\sqrt{2}(\pm 1 \pm i)t}$ , where the choice of signs in both exponentials has to be the same, into the original 2nd-order system and find all such pairs (not that the constants will be determined up to scaling).

This gives (after some algebra which I omit - O.H.) the following four vectors:

$$\mathbf{Answer} : \begin{bmatrix} 4i \\ 1 \end{bmatrix} e^{\sqrt{2}(1+i)t}, \quad \begin{bmatrix} -4i \\ 1 \end{bmatrix} e^{\sqrt{2}(1-i)t}, \quad \begin{bmatrix} -4i \\ 1 \end{bmatrix} e^{\sqrt{2}(-1+i)t}, \quad \begin{bmatrix} 4i \\ 1 \end{bmatrix} e^{\sqrt{2}(-1-i)t}.$$

If you don't like complex-valued functions, you can replace those by their linear combinations which are real-valued, for example, the sum and the difference of the first two and the sum and the difference of the second two. This will give you an alternative answer:

$$\mathbf{Answer} : \begin{bmatrix} -4 \sin \sqrt{2}t \\ \cos \sqrt{2}t \end{bmatrix} e^{\sqrt{2}t}, \quad \begin{bmatrix} 4 \cos \sqrt{2}t \\ \sin \sqrt{2}t \end{bmatrix} e^{\sqrt{2}t}, \quad \begin{bmatrix} 4 \sin \sqrt{2}t \\ \cos \sqrt{2}t \end{bmatrix} e^{-\sqrt{2}t}, \quad \begin{bmatrix} -4 \cos \sqrt{2}t \\ \sin \sqrt{2}t \end{bmatrix} e^{-\sqrt{2}t}.$$

8. Compute the Fourier series of  $e^x$  on the interval  $[-\pi, \pi]$ .

**Solution.** Let us first obtain the exponential Fourier series  $e^x = \sum_{n=-\infty}^{\infty} a_n e^{inx}$  and then convert it to the sine and cosine series. The coefficients  $a_n$  are found from the formula

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-inx} dx = \frac{1}{2\pi} \frac{e^{(1-in)x}}{1-in} \Big|_{-\pi}^{\pi} = \frac{(-1)^n (e^{\pi} - e^{-\pi})}{2\pi(1-in)}.$$

Hence

$$e^x = \frac{e^{\pi} - e^{-\pi}}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(1-in)} e^{inx}.$$

Note that this is a complete answer to the problem. However, if a different form, in terms of sines and cosines, is desirable for some reason, then the conversion is as follows: combine each pair of terms with indices  $n(\neq 0)$  and  $-n$ . This gives

$$\frac{(e^{\pi} - e^{-\pi})(-1)^n}{2\pi} \left( \frac{e^{inx}}{1-in} + \frac{e^{-inx}}{1+in} \right) = \frac{(e^{\pi} - e^{-\pi})(-1)^n (2 \cos nx - 2n \sin nx)}{2\pi(1+n^2)}.$$

$$\mathbf{Answer:} \quad e^x = \frac{e^{\pi} - e^{-\pi}}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(1-in)} e^{inx} = \frac{e^{\pi} - e^{-\pi}}{2\pi} + \sum_{n=1}^{\infty} \frac{(e^{\pi} - e^{-\pi})(-1)^n (2 \cos nx - 2n \sin nx)}{2\pi(1+n^2)}.$$

9. (a) Determine all solutions of the form  $u(x, t) = X(x)T(t)$  to the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \pi, \quad t > 0,$$

that satisfy the boundary conditions

$$u(0, t) = 0, \quad u(\pi, t) + \frac{\partial u}{\partial x}(\pi, t) = 0, \quad t > 0.$$

(b) Describe how to obtain, given a function  $f(x)$ , a solution to the boundary value problem (a) that also satisfies the initial condition

$$u(x, 0) = f(x), \quad 0 < x < \pi.$$

**Solution:** (a) Substituting  $u(x, t) = X(x)T(t)$  leads to the separation condition

$$\frac{X''}{X} = \frac{T'}{T} = K$$

with  $K$  an absolute constant, and the boundary value condition gives  $X(0) = 0$ ,  $X'(\pi) + X(\pi) = 0$ . We need to consider three cases:  $K$  positive, negative or zero.

If  $K = 0$ , then  $X$  must be a linear function passing through the origin but then  $X'(\pi) + X(\pi) = 0$  implies that its leading coefficient must be zero as well. As we need nontrivial solutions, this rules out  $K = 0$ .

If  $K$  is positive, then  $X$  is of the form  $c_1 e^{\sqrt{K}x} + c_2 e^{-\sqrt{K}x}$ , and the boundary value condition becomes

$$c_1 + c_2 = 0, \quad (\sqrt{K} + 1)e^{\sqrt{K}\pi}c_1 + (-\sqrt{K} + 1)e^{-\sqrt{K}\pi}c_2 = 0,$$

and this linear system has only the trivial solution since the determinant of its matrix,  $(-\sqrt{K} + 1)e^{-\sqrt{K}\pi} - (\sqrt{K} + 1)e^{\sqrt{K}\pi}$ , is nonzero. For this to be zero, we must have  $e^{2\sqrt{K}\pi} = (-\sqrt{K} + 1)/(\sqrt{K} + 1)$ , where the expression on the right is less than 1 and the exponential on the left is greater than 1, which is impossible.

Hence  $K$  must be negative, and  $X$  must be of the form  $c_1 \cos \sqrt{-K}x + c_2 \sin \sqrt{-K}x$ . The first boundary condition gives  $c_1 = 0$  and the second becomes  $c_2(\sin \sqrt{-K}\pi + \sqrt{-K} \cos \sqrt{-K}\pi) = 0$ , i.e.,  $c_2$  is arbitrary and

$$\tan \sqrt{-K}\pi = -\sqrt{-K}. \quad (2)$$

The corresponding  $T$  has the form  $ae^{Kt}$ . So, the **answer** is

$$u(x, t) = X(x)T(t) = c_K e^{Kt} \sin \sqrt{-K}x$$

where  $K$  is any one of the solutions to the equation (2).

(b) To match the initial condition  $u(x, 0) = f(x)$ , we must find a way to represent a given function as a series

$$f(x) = \sum_K c_K e^{Kt} \sin \sqrt{-K}x,$$

where the infinite sum is over all solutions to the equation (2). This is not a Fourier series (so we would have to come up with a new method to perform such a decomposition – but you are not asked to do that).