MATH 54, final test.

Name :

Student ID # :

GSI name :

Discussion meeting time :

Problem 1		Problem 6	
Problem 2		Problem 7	
Problem 3		Problem 8	
Problem 4		Problem 9	
Problem 5		Total	

All the necessary work to justify an answer and all the necessary steps of a proof must be shown clearly to obtain full credit. Partial credit **may** be given but only for significant progress towards a solution. 2 pages of notes are allowed. **Books and electronic devices are not allowed during the test.**

If you have taken **only the linear algebra** part of this course, solve **only** problems 1, 2, 3 (labeled LA). Your test length is 1 hour.

If you have taken **only the differential equations** part of this course, solve **only** problems 4, 5, 6, 7, 8, 9 (labeled DE). Your test length is 2 hours.

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1. (LA) 6p. Find the solution set of the linear system

$$\begin{aligned} x_1 - x_2 \qquad\quad - x_4 &= 2 \\ 2x_1 - x_2 + 2x_3 - x_4 &= 6 \\ x_1 - x_2 + 3x_3 + 2x_4 &= 4. \end{aligned}$$

2. (LA) 10p. Consider $L : \mathbb{P}_2 \to \mathbb{P}_2$ where L is the linear transformation defined by

$$L[p(x)] := xp''(x) - p'(x)$$

and $\mathbb{P}_2$ is, as usual, the space of polynomials of degree at most 2. Find a basis for $\ker L$.

3. (LA) 8p. Determine whether the matrix A is diagonalizable:

$$A = \begin{bmatrix} 3 & -1 & 1 \\ 1 & 5 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

4. (DE) 8p. Find a general solution to the differential equation

$$y'' + 2y' + y = e^{-t}.$$

5. (DE) 10p. Reduce the system of ODEs

$$\begin{aligned} \frac{d^2x}{dt^2} - x + 5y &= 0 \\ 2x + \frac{d^2y}{dt^2} + 2y &= 0 \end{aligned}$$

(a) to a system of 1st-order ODEs (do not solve);

(b) to one 4th-order ODE (do not solve).

6. (DE) 8p. Find a general solution to the differential equation

$$y^{(4)} + 4y = 5\cos x.$$

7. (DE) 10p. Find a linear system of ODEs $\vec{x}'(t) = A\vec{x}(t)$ for which the vector functions

$$\begin{bmatrix} e^{-t} \\ 2e^{-t} \\ e^{-t} \end{bmatrix}, \quad \begin{bmatrix} e^{t} \\ 0 \\ e^{t} \end{bmatrix}, \quad \begin{bmatrix} e^{3t} \\ -e^{3t} \\ 2e^{3t} \end{bmatrix}$$

form a fundamental solution set. Justify your answer.

8. (DE) 10p. (a) Show that the Fourier series for $f(x) = x$ on $-\pi \leq x \leq \pi$ is

$$f(x) \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin(nx)$$

(b) Use the result of part (a) to show that

$$\frac{\pi}{4} = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{2k-1} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

9. (DE) 10p. (a) Describe, with proof, all nontrivial solutions of the form

$$u(x,y,t) = X(x)Y(y)T(t)$$

to the boundary-value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, & 0 < x < \pi,\ 0 < y < \pi,\ t > 0 \\ \frac{\partial u}{\partial x}(0,y,t) &= \frac{\partial u}{\partial x}(\pi,y,t) = 0, & 0 < y < \pi,\ t > 0 \\ u(x,0,t) &= u(x,\pi,t) = 0, & 0 < x < \pi,\ t > 0. \end{aligned}$$

(b) Find a linear combination of solutions to part (a) that also satisfies the initial condition $u(x,y,0) = \cos 6x \sin 4y - 3\cos x \sin 11y$.