MATH 202A, FALL 2013.
Homework assignment # 1.

Recommended reading for this homework: Sections 0.1, 0.2, 0.3, 0.4, and beginning of 0.6 of Chapter 0, Prologue of Folland’s “Real analysis”. For more background on set theory, see Halmos’ “Naive set theory” and/or Rademacher and Toeplitz “The enjoyment of math”.

1. Derive the Hausdorff maximal principle from the Axiom of Choice.

2. Prove: If \( A \) is a collection of convex sets that is totally ordered by set inclusion, then the union \( \bigcup_{A \in A} A \) is convex. [Assume addition and multiplication by real scalars is defined for all sets in this problem, and take the usual definition of convexity: a set is convex if it contains the segment \( \alpha x + (1 - \alpha)y, \alpha \in [0,1] \), whenever it contains points \( x \) and \( y \).]

3. A well-known countable set provides an example \( S \) of an interesting variation on the set \( \Omega \) from Proposition 0.18:

\[
S = \bigcup_{\alpha \in A} I_\alpha,
\]

where \( A \) is uncountable, both \( S \) and \( A \) are totally ordered, and the sets \( I_\alpha \) are initial-segment-like, that is \( I_\alpha \subsetneq I_\beta \) and \( x < y \) for any \( x \in I_\alpha \), \( y \in I_\beta \setminus I_\alpha \) for all \( \alpha < \beta \). What is \( S \)? Explain.

4. Let \( C \) be the set of all complex continuous functions on \([0,1]\). Does the function

\[
\rho(f,g) := \int_0^1 \frac{|f(x) - g(x)|}{1 + |f(x) - g(x)|} \, dx
\]

define a metric on \( C \)?

5. Consider the set \( S \) of all numbers between 0 and 1 whose base-5 expansion \( 0.a_1a_2a_3\ldots \) does not contain the digit 1. What is the cardinality of \( S \)?

6. Show that the function

\[
\rho(m,n) := \begin{cases} 
0 & \text{if } m = n \\
1 & \text{if } m \neq n 
\end{cases}
\]

is a metric on \( \mathbb{Z} \). Which sets are open and which are closed in that metric? Is \((\mathbb{Z}, \rho)\) complete?