Solutions to Homework # 4.

1. Prove that, for any positive integer $n$,

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

**Solution.** Expand both sides of the identity

$$(1 + x)^n(1 + x)^n = (1 + x)^{2n},$$

using the binomial formula. We get

$$\sum_{j=0}^{n} \binom{n}{j} x^j \sum_{k=0}^{n} \binom{n}{k} x^k = \sum_{\ell=0}^{2n} \binom{n}{\ell} x^\ell.$$  

The coefficient of $x^n$ in the left-hand side is equal to

$$\sum_{j+k=n} \binom{n}{j} \binom{n}{k} = \sum_{j=0}^{n} \binom{n}{j} \binom{n}{n-j} = \sum_{j=0}^{n} \binom{n}{j}^2,$$

whereas the same coefficient on the right is $\binom{2n}{n}$. Thus

$$\sum_{j=0}^{n} \binom{n}{j}^2 = \binom{2n}{n}.$$

2. Starting with 0, two players alternately add 1, 2, or 3 to a single running total. The player who first brings the total to at least 1000 wins. Prove that the second player has a strategy to win against any strategy for the first player.

**Solution.** The winning strategy of the second player is to bring the running total to a multiple of 4. No matter how the first player starts (say, with $n_1$ between 1 and 3), the second player will respond with $4 - n_1$, and subsequently with $4 - n_2$ in response to $n_2$ etc. At the second-to-last step, the first player brings the total to a number between 997 and 999, and the second player obtains 1000.

3. At first, a room is empty. Every minute, either one person enters or two people leave. After exactly $3^{1999}$ minutes, could the room contain $3^{1000} + 2$ people?

**Solution.** Note that the difference between any two possible numbers of people in the room at any given moment is a multiple of 3. One possible number of people after exactly $3^{1999}$ is $3^{1999}$ people (if one person enters every minute). That number is a multiple of 3, but $3^{1000} + 2$ is not. So, the room cannot contain $3^{1000} + 2$ people.

4. Evaluate

$$\int_{0}^{\pi/2} \frac{dx}{1 + (\tan x)^{\sqrt{2}}}.$$
Solution. Rewrite the integral as
\[ I = \int_0^{\pi/2} \frac{(\cos x)\sqrt{2}}{(\cos x)\sqrt{2} + (\sin x)\sqrt{2}} \, dx \]
and perform the substitution \( y = \pi/2 - x \). The integral becomes
\[ I = \int_0^{\pi/2} \frac{(\sin y)\sqrt{2}}{(\sin y)\sqrt{2} + (\cos y)\sqrt{2}} \, dy. \]
Adding these two expressions, we get
\[ 2I = \int_0^{\pi/2} \frac{(\cos y)\sqrt{2} + (\sin y)\sqrt{2})\, dy}{(\sin y)\sqrt{2} + (\cos y)\sqrt{2}} = \int_0^{\pi/2} \, dy = \pi/2, \]
hence the value of the integral is \( I = \pi/4 \).

5. Show that \( n^4 - 20n^2 + 4 \) is composite when \( n \) is an integer.

Solution. Note that the polynomial is a difference of squares \((n^2 - 2)^2 - 16n^2\), hence factors as
\[ (n^2 - 4n - 2)(n^2 + 4n - 2). \]
We need to rule out the possibility that this factorization is trivial, i.e., one of the factors is \( \pm 1 \), and the other factor is prime. Testing the possibilities \( n^2 \pm 4n - 2 = \pm 1 \), we see that none of these quadratic equations has an integer solution. This shows that the above factorization is always nontrivial, so the resulting number is composite.

6. A restaurant gives one of five types of coupons with each meal, each with equal probability. A customer receives a free meal after collecting one coupon of each type. How many meals does a customer expect to need to buy before getting a free meal?

Solution. Let \( X \) be the random variable corresponding to the number of tries before collecting coupons of all five types. The problem asks to determine the expected value \( E(X) \) of \( X \). Note that \( X = X_1 + \cdots + X_5 \), where \( X_j, j = 1, \ldots, 5 \), denotes the number of tries to get a coupon of type \( j \). By the linearity of expectation,
\[ E(X) = \sum_{j=1}^5 E(X_j). \]
The probability of collecting the \( j \)th coupon after \( j - 1 \) distinct coupons have been collected is \( p_j = (n - (j - 1))/n \). So, each \( X_j \) is geometrically distributed with \( E(X_j) = 1/p_j \). Hence
\[ E(X) = \sum_{j=1}^5 \frac{5}{5 - (j - 1)} = 11 \frac{5}{12}. \]

7. Someone writes \( n \) letters and writes the corresponding addresses on \( n \) envelopes. How many different ways are there of placing all the letters in the wrong envelopes, each envelope containing exactly one letter?
**Solution.** Let us count the number $N$ of ways to put at least one letter into the right envelope. Then $n! - N$ is the answer to our question. Let $A_i$ denote all possible placements of letters whereby the $i$th letter is in the correct $i$th envelope. By the inclusion-exclusion principle,

$$N = |A_1 \cup \cdots \cup A_n| = \sum_{j=1}^{n} (-1)^{j-1} \sum_{K \subseteq [n], |K| = j} |\cap_{k \in K} A_k|,$$

where $[n]$ denotes the set $\{1, \ldots, n\}$. Note that

$$\sum_{K \subseteq [n], |K| = j} |\cap_{k \in K} A_k| = \binom{n}{j} (n-j)! = \frac{n!}{j!},$$

which corresponds to choosing $j$ letters and placing them into correct envelopes, while the other letters can go anywhere in the remaining envelopes. So, the answer is

$$n! - N = n! + \sum_{j=1}^{n} (-1)^{j} \frac{n!}{j!} = \sum_{j=0}^{n} (-1)^{j} \frac{n!}{j!}.$$ 

8. Parallel lines are drawn at intervals $d$ on a table. A needle of length $1(< d)$ is thrown at random on the table. What is the probability that the needle will touch one of the parallels?

**Solution.** Consider the lines as drawn vertically on the table. The needle falls at a certain angle $\theta \in [0, \pi]$ to the horizontal, and with probability one the lower endpoint of the needle falls in between two vertical lines. The horizontal projection of the needle has length $\cos \theta$, and the probability of the needle hitting one line is proportional to its horizontal projection and inversely proportional to the distance $d$ between vertical lines. Averaging over all possible angles $0$ to $\pi$ gives the desired probability $p$

$$p = \frac{1}{\pi} \int_{0}^{\pi} \frac{\cos \theta}{d} \, d\theta = \frac{2}{\pi d}.$$