Solutions to Homework # 3.

1. Without a calculator, determine what is larger
   \[ \pi^e \quad \text{or} \quad e^\pi. \]

**Solution.** Consider the function \( f(x) := \ln x / x \) on the positive axis \( x > 0 \). As \( f'(x) = 1 / x^2 - \ln x / x^2 \), we see that \( f \) is increasing on the interval \((0, e)\) and decreasing on \((e, \infty)\), in particular, \( x = e \) is the global maximum of that function. Hence
   \[ \frac{\ln e}{e} > \frac{\ln \pi}{\pi}, \quad \text{i.e.,} \quad e \ln \pi < \pi \quad \text{i.e.,} \quad \pi^e = e^{\ln \pi} < e^\pi. \]

2. A natural number is *perfect* if it is the sum of its divisors smaller than itself; for example, 6 and 28 are perfect. Prove that if \( 2^n - 1 \) is prime, then \( 2^{n-1}(2^n - 1) \) is perfect.

**Solution.** The (proper) divisors of \( 2^{n-1}(2^n - 1) \) are all powers of 2 starting from 1 and ending with \( 2^{n-1} \) and the same powers of 2 multiplied by \( 2^{n-1} - 1 \), with the exception of the number \( 2^{n-1}(2^n - 1) \) itself. This gives the sum
   \[ 1 + 2 + \cdots + 2^{n-1} + (1 + 2 + \cdots + 2^{n-2})(2^{n-1} - 1) = 2^n - 1 + (2^{n-1} - 1)(2^n - 1) = (2^n - 1)2^{n-1}. \]

3. The Nelsons have gone out for the evening, leaving their four children with a new babysitter, Nancy Wiggens. Among the many instructions the Nelsons gave Nancy before they left was that three of their children were consistent liars and only one of them consistently told the truth. But in the course of receiving so much other information, Nancy forgot which one was the truar. As she was preparing dinner for the children, one of them broke a vase in the next room. Nancy asked who broke the vase. These were the children’s statements:

   - **Betty:** Steve broke the vase.
   - **Steve:** John broke it.
   - **Laura:** I didn’t break it.
   - **John:** Steve lied when he said I broke it.

Knowing that only one of these statements was true, Nancy quickly determined which child broke the vase. Who was it?

**Solution.** Note that two statements here, of Steve and of John, are exact opposites, so one of them is true. If John broke the vase, then also Laura’s statement is true, which is impossible. Hence Laura’s statement is false: she broke the vase.

4. Suppose that the points in a tennis game are independent and that the server wins each point with probability \( p \). The first player to have at least four points and at least two points more than the opponent wins the game. What is the probability that the server wins the game?
Solution. The server wins each particular point with probability $p$; denote $1 - p$ by $q$. First, consider the ways the server can win with exactly four points. The other player may score zero, one or two points before the server scores four, and the total probability for these mutually exclusive possibilities is

$$p^4 + \binom{4}{1}p^4q + \binom{5}{2}p^4q^2.$$ 

The server may also win after the game reaches a $3-3$ tie; a $3-3$ tie happens with probability $\binom{6}{3}p^3q^3$. Afterwards, the probability that the server wins after a total of exactly $2k + 2$ more points is $(2pq)^k p^2$. Summing this over all $k \geq 0$ yields the geometric series $p^2 \sum_{k=0}^{\infty} (2pq)^k = p^2/(1 - 2pq)$. Hence the total probability is

$$p^4(1 + 4q + 10q^2) + \frac{20p^5q^3}{1 - 2pq}.$$ 

Remark. This equals .736 when $p = .6$ and .901 when $p = 7$; this suggests the difficulty of “breaking serve”.

5. Show that every graph contains two vertices of equal degree. (The degree of a vertex is the number of edges adjacent to it.)

Solution. Let a graph have $n$ vertices. The degree of a vertex can be anywhere between 0 and $n - 1$. The numbers 0 and $n - 1$ cannot both occur as degrees in the same graph, since this would mean that there is a vertex connected to every other vertex and another vertex not connected to any vertex. Thus the possible degrees are either 1 through $n - 1$ or 0 through $n - 2$. In either case, the total number of possibilities is at most $n - 1$, and the number of vertices is $n$, hence the pigeonhole principle implies that there are at least two vertices with the same degree.

Remark: Notice that this problem requires the understanding that the graph is simple, i.e., without loops and multiple edges.

6. Given a circle of $n$ lights, exactly one of which is initially on, it is permitted to change the state of a bulb provided one also changes the state of every $d$th bulb after it (where $d$ is a divisor of $n$ strictly less than $n$), provided that all $n/d$ bulbs were originally in the same state as one another. For what values of $n$ is it possible to turn on all bulbs via a series of moves of this kind?

Solution. Place each light on the unit circle located at an $n$th root of unity. Thus the lights are in positions $1, \omega, \ldots, \omega^{n-1}$, where $\omega := e^{2\pi i/n}$.

Without loss of generality, we can assume that the light at 1 is initially on, whereas the others are off. Now, if $d < n$ is a divisor of $n$ and the lights at

$$\omega^a, \omega^{a+d}, \omega^{a+2d}, \ldots, \omega^{a+(n/d-1)d}$$
are in the same state, then we can change the state of these \( n/d \) lights. But the sum of these is
\[
\omega^n(1 + \omega^d + \cdots + \omega^{(n/d-1)d}) = \omega^n(\omega^n - 1)/(\omega^d - 1) = 0.
\]
This fact shows that if we add up all the roots of unity with the light “on”, the sum will never change, since whenever we make an allowable change of state, they add up to zero. The original sum was equal to 1, but if all lights are on, then the sum will be
\[
1 + \omega + \cdots + \omega^{n-1} = 0 \neq 1,
\]
so we can never turn on all the lights.

7. A large house contains a TV set in each room that has an odd number of doors. There is only one entrance to the house. Show that it is always possible to enter this house and get to a room with a TV.

Solution. Consider the house as a graph where each vertex represents a room, and each edge represents a door between two rooms. An additional vertex will represent the area outside the house (connected to one of the room though the entrance to the house). Without loss of generality, we can consider only the rooms connected to the outside, i.e., the connected component of this graph with the edge representing the entrance.

We know that the sum of the degrees of all vertices is even in any graph (being twice the number of edges). Since there is just one entrance to the house, the degree of the vertex representing the outside area is 1. Therefore, the sum of the degrees of all the vertices representing rooms connected to the entrance is odd. Therefore there is at least one room with an odd number of doors connected to the entrance. By the assumption of the problem, such a room has a TV. So it is always possible to reach a room with a TV.

8. Let \( f(n) = n + \lfloor \sqrt{n} \rfloor \). Prove that, for every positive integer \( m \), the sequence
\[
m, f(m), f(f(m)), f(f(f(m))), \ldots
\]
contains a perfect square (i.e., a square of an integer).

Solution. Consider the “remainder” \( r(n) = n - \lfloor \sqrt{n} \rfloor^2 \). We show that one or two applications of the function \( f \) will always decrease the remainder. This will show that the function \( f \) can be applied enough time to yield remainder zero, which means exactly that the number itself will be a perfect square.

Indeed, suppose \( n \) is not a perfect square. Denoting \( \lfloor \sqrt{n} \rfloor \) by \( m \), we are in one of the two cases:

(A) \( m^2 < n \leq m^2 + m \) \quad or \quad (B) \( m^2 + m < n \leq m^2 + 2m \),

since the next perfect square is \( m^2 + 2m + 1 \). In case (A), we get \( m^2 + m < f(n) = n + m \leq m^2 + 2m \) and one more application of \( f \) gives
\[
m^2 + 2m < f(f(n)) = n + 2m \leq m^2 + 3m, \quad \text{hence} \quad (m + 1)^2 \leq f(f(n)) < (m + 2)^2,
\]
hence \( r(f(f(n))) = n + 2m - (m + 1)^2 = n - m^2 - 1 = r(n) - 1 \). So, the remainder goes down by 1. In case (B), the remainder goes down by more than 1 since \( m^2 + 2m < f(n) = n + m \leq m^2 + 3m \), hence \( r(f(n)) = f(n) - (m+1)^2 = n + m - (m+1)^2 = n - m^2 - m - 1 = r(n) - m - 1 \).