1. Prove the identity

\[ \sum_k \binom{m}{k} \binom{n+k}{m} = \sum_j \binom{m}{j} \binom{n}{j} 2^j. \]

**Solution.** Note that both parts count the number of ordered pairs \((X,Y)\) where \(X \subseteq A, Y \subseteq B \cup X\), and \(|Y| = m\), where \(A\) and \(B\) are two disjoint sets with \(|A| = m\) and \(|B| = n\). Indeed, the left-hand side corresponds to the usual counting where, for every \(k\), we first choose \(k\) out of \(m\) elements of \(A\) to form \(X\) and then \(m\) elements out of \(n+k\) elements of \(B \cup X\) to form \(Y\). The right-hand side corresponds to the following: let \(Y_1 := Y \cap B\) and \(Y_2 := Y \cap X\). Then \(Y_1 \subseteq B, Y_1 \subseteq A, |Y_1| + |Y_2| = m\), and \(Y_2 \subseteq X\). For all \(j\), consider the situation \(|Y_1| = j, |Y_2| = m-j\). There are

\[ \binom{n}{j} \cdot \frac{m}{m-j} = \binom{n}{j} \cdot \frac{m}{j} \]

ways to choose the pair \(Y_1\) and \(Y_2\). Moreover, \(X\) can contain any elements of the remaining set \(A \setminus Y_2\), where \(|A \setminus Y_2| = j\). So, the number of ways to form \(X\) is \(2^j\). This gives the right-hand side total

\[ \sum_{j=0}^{m} \binom{n}{j} \binom{m}{j} 2^j. \]

2. Evaluate the limit

\[ \lim_{n \to \infty} n \sin(2\pi en!). \]

**Solution.** Using the series for \(e\) and the \(2\pi\)-periodicity of \(\sin x\), we obtain

\[ n \sin(2\pi en!) = n \sin \left(2\pi n! \sum_{k=0}^{\infty} \frac{1}{k!}\right) = n \sin \left(2\pi \sum_{k=1}^{\infty} \frac{1}{(n+1) \cdots (n+k)}\right) = \frac{2\pi n}{n+1} + O \left(\frac{1}{n}\right) \to 2\pi \text{ as } n \to \infty. \]

3. Prove that the gcd of two positive integers \(a\) and \(b\) can be represented as the double sum

\[ \gcd(a,b) = \sum_{m=0}^{a-1} \sum_{n=0}^{b-1} \frac{1}{a} e^{2\pi ibmn/a}. \]

**Solution.** Let \(d\) denote the gcd of \(a\) and \(b\). Then \(a = rd, b = sd\) for some relatively prime integers \(r\) and \(s\). Moreover, the product \((b/a)m = (s/r)m\) takes integer values exactly for \(d\) values of \(m\), namely, \(m = 0, r, 2r, (d-1)r\). If \((b/a)m\) is not an integer, then

\[ \sum_{n=0}^{a-1} e^{2\pi ibmn/a} = 0. \]
If \((b/a)m\) is an integer, then the same sum is equal to \(a\) since each of its \(a\) terms equals 1. Hence

\[
\sum_{m=0}^{a-1} \sum_{n=0}^{a-1} \frac{1}{a} e^{2\pi i bmn/a} = \frac{1}{a} da = d.
\]

4. Find the number of ways to divide a convex \((n + 2)\)-gon into triangles by diagonals that do not intersect in the interior of the polygon.

**Solution.** Let \(a_n\) denote the number of ways to divide a convex \((n+2)\)-gon into triangles as specified. By convention, we take \(a_0 = a_1 = 1\). The next few values are \(a_2 = 2, a_3 = 5, a_4 = 14\), which are the first Catalan numbers. To prove that, in general, \(a_n\) is the \(n\)th Catalan number, we need to demonstrate that the \(a_n\)'s satisfy the recurrence relation that defines the Catalan numbers. Indeed, number its vertices 0 through \(n + 1\) consecutively. Consider the side joining vertices \(n\) and \(n + 1\). If the triangle containing this side has vertex \(k\) as its remaining point, then the diagonals joining vertices \(n\) and \(n + 1\) to vertex \(k\) define, in addition to the given triangle, a convex \((k+2)\)-gon and a convex \((n+1−k)\)-gon. There are \(a_k a_{n−k−1}\) ways to finish the division of the original polygon into triangles. Summing over \(k\), we obtain

\[
a_n = \sum_{k=0}^{n-1} a_k a_{n−k−1}.
\]

The Catalan numbers

\[
\frac{1}{n+1} \binom{2n}{n}
\]

satisfy the recurrence, hence

\[
a_n = \frac{1}{n+1} \binom{2n}{n}.
\]

5. If \(s_n\) denotes the sum of the first \(n\) positive integers, find the sum of the infinite series

\[
S = \sum_{n=1}^{\infty} \frac{s_n}{2^n-1}.
\]

**Solution.** Since \(s_n = n(n+1)/2\), we have

\[
S = \sum_{n=1}^{\infty} \frac{n(n+1)}{2^n} = \left( \frac{d}{dx} \sum_{n=1}^{\infty} nx^{n+1} \right) \bigg|_{x=1/2} = \left( \frac{d}{dx} x^2 \sum_{n=0}^{\infty} (n+1)x^n \right) \bigg|_{x=1/2} = \left( \frac{d}{dx} \frac{x^2}{(1-x)^2} \right) \bigg|_{x=1/2} = 4 + 4 = 8.
\]

6. The squares of an \(n \times n\) chessboard \((n \geq 2)\) are labeled 1, 2, \ldots, \(n^2\) (every number occurs exactly once). Prove that there exist two neighboring squares (i.e., squares that share a common edge) such that their labels differ by at least \(n\).
Solution. Suppose the labels of any two neighboring squares differ by at most \( n - 1 \). For 
\( k = 1, 2, \ldots, n^2 - n \), let \( A_k \), \( B_k \) and \( C_k \) denote the sets of squares labelled by 1 through \( k \); 
of squares labelled by \( k + 1 \) through \( k + n - 1 \); and of squares labelled by \( k + n \) through 
\( n^2 \); respectively. By the assumption, the squares from \( A_k \) and \( C_k \) have no edge in common, 
and \( B_k \) contains \( n - 1 \) elements only. Consequently, for each \( k \) there is an entire row and an 
entire column of squares belonging either to \( A_k \) or \( C_k \).

For \( k = 1 \), they must belong to \( C_k \) since \( A_k \) has too few elements, while for \( k = n^2 - n \) 
they must belong to \( A_k \). Let \( k \) be the smallest index for which \( A_k \) contains an entire row 
and an entire column. Since \( C_{k-1} \) has that property too, it must have at least two squares 
common with \( A_k \), which is impossible. This proves the result by contraction.

7. Let \( a_1 \geq a_2 \geq \cdots \geq a_n \) be real numbers such that
\[
a_1^k + a_2^k + \cdots + a_n^k \geq 0
\]
for all integers \( k > 0 \). Let \( p := \max\{|a_1|, \ldots, |a_n|\} \). Prove that \( p = a_1 \) and that
\[
(x - a_1)(x - a_2)\cdots(x - a_n) \leq x^n - a_1^n \quad \text{for all } x > a_1.
\]

Solution. First of all, \( a_1 > 0 \), and if \( p \neq a_1 \), we must have \( a_n < 0 \), \( |a_n| > |a_1| \), 
and \( p = -a_n \).

But then, for sufficiently large odd \( k \), \(-a_n^k = |a_n|^k > (n - 1)|a_1|^k\), so that
\[
a_1^k + \cdots + a_n^k \leq (n - 1)|a_1|^k - |a_n|^k < 0,
\]
which is a contradiction. Hence \( p = a_1 \).

Now let \( x > a_1 \). From \( a_1 + \cdots + a_n \geq 0 \) we deduce
\[
\sum_{j=2}^{n} (x - a_j) \leq (n - 1) \left(x + \frac{a_1}{n-1}\right),
\]
so by the arithmetic-geometric inequality,
\[
(x - a_2)\cdots(x - a_n) \leq \left(x + \frac{a_1}{n-1}\right)^{n-1} \leq x^{n-1} + x^{n-2}a_1 + \cdots + a_1^{n-1}. \tag{1}
\]
The last inequality here holds because \( \binom{n-1}{r} \leq (n - 1)^r \) for all \( r \geq 0 \). Multiplying (1) by 
\( x - a_1 \) yields the desired inequality.

8. The following operation is allowed on a finite graph: Choose an arbitrary cycle of length 4 
(if there is any), choose an arbitrary edge in that cycle, and delete it from the graph. For 
a fixed integer \( n \geq 4 \), find the least number of edges of a graph that can be obtained 
by repeated applications of this operation from the complete graph on \( n \) vertices (i.e., the graph 
where each pair of vertices is joined by an edge).

Solution. The least number of edges in such a graph is \( n \). Here is a proof. We first note that 
deleting edge \( AB \) in a 4-cycle \( ABCD \) from a connected an nonbipartite graph \( G \) yields a 
connected and a nonbipartite graph, say \( H \). The connectedness is obvious, since all vertices 
\( ABCD \) are still connected to each other. Also, if \( H \) were bipartite with partition of the set

3
of vertices into $P_1$ and $P_2$, then without loss $A, C \in P_1$ and $B, D \in P_2$, so $G = H \cup AB$ would also be bipartite with the same partition, a contradiction.

Any graph that can be obtained from the complete $n$-graph in the described fashion is therefore connected and has at least 1 cycle (otherwise it would be bipartite), hence it must have at least $n$ edges.

Now we show that we can always obtain a graph with exactly $n$ edges. Label the vertices $v_1$ through $v_n$. Remove edges $v_iv_j$ for all $3 \leq i < j < n$ from each cycle $v_2v_i v_j v_n$. Then, for $i = 3, \ldots, n - 1$, remove edges $v_2v_i$ and $v_iv_n$ from the cycles $v_1v_i v_2 v_n$ and $v_1 v_i v_n v_2$ respectively, thus obtaining a graph with exactly $n$ edges: $v_1v_i$ ($i = 2, \ldots, n$) and $v_2 v_n$. 