1. Let $x, y$ be positive real numbers. Prove that
\[ x^y + y^x > 1. \]

**Solution.** If $x \geq 1$, then $x^y + y^x > x^y \geq 1$. If $0 < x < 1$, then $\frac{1}{x^y} < \left(1 + \frac{1}{y}\right)^x < 1 + \frac{x}{y}$ by Bernoulli’s inequality, whence $y^x > \frac{y}{x+y}$. Similarly $x^y > \frac{x}{x+y}$, so $x^y + y^x > \frac{x}{x+y} + \frac{y}{x+y} = 1.$

2. Let $x > 0$. Prove that
\[
\sqrt{1 + x \sqrt{1 + (x+1) \sqrt{1 + (x+2) \sqrt{1 + \cdots}}}} = x + 1.
\]

**Solution.** Let $f(x) = \sqrt{1 + x \sqrt{1 + (x+1) \sqrt{1 + (x+2) \sqrt{1 + \cdots}}}}$. Note that $f(x)$ satisfies the functional equation
\[ f(x)^2 = 1 + xf(x+1). \]

We have
\[ f(x) \geq \sqrt{x \sqrt{x \cdots}} = x^{1/2 + 1/4 + \cdots} = x \]
and
\[
\begin{align*}
 f(x) &\leq \sqrt{(x+1) \sqrt{(x+2) \sqrt{\cdots}}} \\
 &\leq \sqrt{(x+1) \sqrt{2(x+1) \sqrt{3(x+1) \cdots}}} \\
 &= (x+1) \sqrt{1 \sqrt{2 \sqrt{3 \cdots}}} \\
 &\leq (x+1) \sqrt{1 \sqrt{2 \sqrt{4 \sqrt{8 \cdots}}}} \\
 &= 2(x+1).
\end{align*}
\]

Therefore $\frac{x+1}{2} \leq f(x) \leq 2(x+1)$ for all $x \geq 1$. Given $A(x+1) < f(x) < B(x+1)$ with $A < 1 < B$, we have
\[ 1 + Ax(x+2) < 1 + xf(x+1) < 1 + Bx(x+2) \]
so
\[
A(x+1)^2 < 1 + Ax(x+2) < f(x)^2 < 1 + Bx(x+2) < B(x+1)^2
\]
\[ \sqrt{A(x+1)} < f(x) < \sqrt{B(x+1)}. \]
By induction, $2^{-2^{-k}}(x + 1) < f(x) < 2^{2^{-k}}(x + 1)$ for all $k \geq 0$, so letting $k \to \infty$ gives $f(x) = x + 1$.

3. Let $a$, $b$ and $c$ be the side lengths of a triangle. Show that

$$\frac{3}{2} \leq \frac{a}{b + c} + \frac{b}{a + c} + \frac{c}{a + b} < 2.$$

**Solution.** Since $a$, $b$, and $c$ measure the sides of a triangle, we have $a + b > c$. Let $s = (a + b + c)/2$; then $a + b > c$ implies $a + b > s$. Hence $\frac{c}{a+b} < \frac{s}{2}$, and, similarly, $\frac{a}{b+c} < \frac{s}{2}$ and $\frac{b}{a+c} < \frac{s}{2}$. Adding these inequalities gives

$$\frac{a}{b + c} + \frac{b}{a + c} + \frac{c}{a + b} < \frac{a+b+c}{s} = 2.$$

On the other hand, assume, without loss of generality, that $a \leq b \leq c$. Then $\frac{1}{b+c} \leq \frac{1}{a+c} \leq \frac{1}{a+b}$. By the rearrangement inequality,

$$\frac{a}{b + c} + \frac{b}{a + c} + \frac{c}{a + b} \geq \frac{a}{a+b} + \frac{b}{b+c} + \frac{c}{a+c}$$

so $2 \left( \frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \right) \geq 3$ by adding these together.

4. Do there exist integers $a$, $b$, $c$ and $d$ such that the polynomial $p(x) = ax^3 + bx^2 + cx + d$ takes values $p(19) = 1$ and $p(62) = 2$?

**Solution.** For any polynomial $p(x) \in \mathbb{Z}[x]$, we have $p(x) - p(y) \in (x - y)\mathbb{Z}[x, y]$, e.g., $x^4 - y^4 = (x - y)(x^3 + x^2y + xy^2 + y^3)$. In our case, for any polynomial with integer coefficients $p(62) - p(19)$ is divisible by $62 - 19 = 43$. Therefore the stated conditions cannot be satisfied.

5. Let $m$ and $n$ be positive integers. Show that $m^2 + n$ or $n^2 + m$ is not a perfect square.

**Solution.** If $m^2 + n$ is a perfect square, then $m^2 + n \geq (m+1)^2$, so $n \geq 2m+1$. Similarly, if $n^2 + m$ is a perfect square, then $m \geq 2n+1$. These two inequalities cannot be simultaneously satisfied by positive integers, since they together imply $n \geq 4n + 3$.

6. Consider a $6 \times 6$ board tiled by $1 \times 2$ dominos. Prove that the board can be cut into two rectangles, each of which is tiled separately.

**Solution.** There are 10 internal lines on the $6 \times 6$ board, 5 horizontal and 5 vertical. Suppose that the board is tiled so that every line is crossed. There must be a domino placed across the first horizontal line. But if a line crosses a domino, it crosses at least two, since there is an even number of dominos on either side. As there are 10 lines, 20 dominos are needed, but only 18 dominos can be placed over the board.
7. The Riemann ζ-function is defined as

\[ \zeta(z) := \sum_{n \in \mathbb{N}} \frac{1}{n^z}. \]

Prove that

\[ \sum_{k=0}^{\infty} \frac{1}{(4k+1)^3} = \frac{\pi^3}{64} + \frac{7}{16} \cdot \zeta(3). \]

**Solution.** From

\[ \sum_{k \geq 1} \frac{1}{(2k)^3} = \frac{1}{8} \zeta(3) \]

it follows that

\[ \sum_{k \geq 0} \left( \frac{1}{(4k+1)^3} + \frac{1}{(4k+3)^3} \right) = \frac{7}{8} \zeta(3) \]

so that we will be done if we prove that

\[ \sum_{k \geq 0} \left( \frac{1}{(4k+1)^3} - \frac{1}{(4k+3)^3} \right) = \frac{\pi^3}{32}. \]

Start with the identity

\[ \pi \cot(\pi x) = \sum_{k=0}^{\infty} \left( \frac{1}{x+k} + \frac{1}{x-k-1} \right) \]

and differentiate twice to get

\[ \frac{d^2}{dx^2} \pi \cot(\pi x) = 2\pi^3 \cot(\pi x) \csc^2(\pi x) = 2 \sum_{k \geq 0} [(x+k)^{-3} + (x-k-1)^{-3}] . \]

Substituting \( x = \frac{3}{4} \) yields

\[ 2\pi^3 = \sum_{k=0}^{\infty} \left( \frac{1}{(k+\frac{1}{4})^3} - \frac{1}{(k+\frac{3}{4})^3} \right) \]

whence the desired identity immediately follows.

8. Show that

\[ \sum_{k=0}^{n} \binom{n}{k} \sin(a+2k)\theta = 2^n \cos^n \theta \cdot \sin(a+n)\theta. \]

**Solution.** The left-hand side is the imaginary part of \( \sum_{k=0}^{n} \binom{n}{k} e^{i(a+2k)\theta} = e^{ia\theta} \sum_{k=0}^{n} \binom{n}{k} e^{2ki\theta} = e^{ia\theta}(1 + e^{2i\theta})^n = e^{ia\theta}(e^{i\theta} \cdot 2 \cos \theta)^n = e^{i(a+n)\theta}(2 \cos \theta)^n. \)