Solutions to homework # 1.

1. A square is partitioned into rectangles whose sides are parallel to the sides of the square. For each rectangle, the ratio of its shorter side to its longer side is determined. Prove that the sum $S$ of these ratios is at least 1.

**Solution.** Suppose that there are $n$ rectangles and that their shorter sides are $a_1, \ldots, a_n$, while their longer sides are $b_1, \ldots, b_n$, respectively. Then

$$S = \sum_{j=1}^{n} \frac{a_j}{b_j}. \quad \text{(1)}$$

Now, if $x$ is the side of the square, then the areas of the rectangles sum up to $x^2$:

$$\sum_{j=1}^{n} a_j b_j = x^2. \quad \text{(2)}$$

Note that no side $b_j$ is bigger than the side $x$ of the square. This implies

$$S = \sum_{j=1}^{n} \frac{a_j}{b_j} = \sum_{j=1}^{n} \frac{a_j b_j}{b_j^2} \geq \sum_{j=1}^{n} \frac{a_j b_j}{x^2} = \frac{1}{x^2} \sum_{j=1}^{n} a_j b_j = \frac{1}{x^2} x^2 = 1.$$

2. If $x_1$ through $x_7$ are real numbers such that

$$
\begin{align*}
    x_1 + 4x_2 + 9x_3 + 16x_4 + 25x_5 + 36x_6 + 49x_7 &= 1, \\
    4x_1 + 9x_2 + 16x_3 + 25x_4 + 36x_5 + 49x_6 + 64x_7 &= 12, \\
    9x_1 + 16x_2 + 25x_3 + 36x_4 + 49x_5 + 64x_6 + 81x_7 &= 123,
\end{align*}
$$

what is the value of

$$S = 16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7?$$

**Solution.** To express $S$ as a linear combination of the left-hand sides of the previous equations, we need to find constants $a$, $b$ and $c$ such that

$$an^2 + b(n+1)^2 + c(n+2)^2 = (n+3)^2$$

for all $n = 1, \ldots, 7$ (hence for all $n$). This is equivalent to solving the linear system

$$\begin{align*}
    a + b + c &= 1, \\
    2b + 4c &= 6, \\
    b + 4c &= 9,
\end{align*}$$

which gives $b = -3$, $c = 3$, and $a = 1$. Therefore, the sum $S$ is equal to

$$a \cdot 1 + b \cdot 12 + c \cdot 123 = 1 - 36 + 369 = 334.$$
3. Find the last three digits of $13^{398}$.

**Solution.** The last three digits are determined by the residue class mod 1000. Since $\phi(1000) = 400$, Euler’s Theorem implies that $13^{400} \equiv 1 \pmod{1000}$. So, we need to solve the equation $13^2x \equiv 1 \pmod{1000}$. By using the Euclidean algorithm, we find that

$$169 \cdot (-71) + 12 \cdot 1000 = 1,$$

so the residue class is that of $-71$, which is the same as $929$. Therefore, the last three digits are $929$.

4. For what values of the variable $x$ does the following inequality hold:

$$\frac{4x^2}{(1 - \sqrt{1 + 2x})^2} < 2x + 9?$$

**Solution.** The left-hand side is defined if $x \geq -1/2$ and $x \neq 0$. If these conditions are satisfied, we may multiply the numerator and the denominator by $(1 + \sqrt{1 + 2x})^2$ and simplify. We thus get the following equivalent problem

$$2\sqrt{2x + 1} < 7, \quad x \geq -1/2, \quad x \neq 0.$$

This last inequality is satisfied if $x \neq 0$ and $-1/2 \leq x < 45/8$.

5. Let $p$ be a prime greater than 3 and let $n = (2^{2p} - 1)/3$. Prove that $n$ divides $2^n - 2$.

**Solution.** Note that

$$n - 1 = 4(2^{p-1} - 1)(2^{p-1} + 1)/3.$$

The factor 4 is divisible by 2, whereas the factor $2^{p-1} - 1$ is divisible by $p$ and by 3 by Fermat’s little theorem since $p$ is an odd prime. Also $p > 3$, so $2^{p-1} - 1$ is divisible by $3p$, and hence $n - 1$ is divisible by $2p$.

By the assumption of the problem, $2^{2p} - 1$ is divisible by $n$. Since $n$ is divisible by $2p$, we conclude that $2^{n-1} - 1$ is divisible by $2^{2p} - 1$. (The latter uses the fact that $a^k - b^k$ divides $a^{mk} - b^{mk}$ for any integers $a$, $b$, $k$, and $m$.) In summary, $2^{n-1} - 1$ is divisible by $2^{2p} - 1$, hence also divisible by $n$, and therefore $2^n - 2$ is also divisible by $n$.

6. Let $n_1 < n_2 < \cdots < n_k < \cdots$ be a sequence of integers such that

$$\lim_{k \to \infty} \frac{n_k}{n_1 n_2 \cdots n_{k-1}} = \infty.$$

Prove that the sum of the series $\sum_{i=1}^{\infty} 1/n_i$ is irrational.

**Solution.** Assume that

$$\sum_{i=1}^{\infty} \frac{1}{n_i} = \frac{p}{q}$$

where $p$ and $q$ are integers. From the limit assumption of the problem, there exists a number $k$ such that

$$\frac{n_k}{n_1 n_2 \cdots n_{k-1}} > 3 \quad \text{and} \quad \frac{n_{i+1}}{n_i} > 3 \quad \text{for all} \quad i \geq k.$$
Then
\[ p_{1} \cdots n_{k-1} = \sum_{i=1}^{k-1} \frac{n_{1} \cdots n_{k-1}q}{n_{i}} + \sum_{i=k}^{\infty} \frac{n_{1} \cdots n_{k-1}q}{n_{i}}. \]

The product \( p_{1} \cdots n_{k-1} \) and the finite sum in the right-hand side about are integers, whereas each term in the infinite sum is bounded as
\[ \frac{n_{1} \cdots n_{k-1}q}{n_{i}} < \left(\frac{1}{3}\right)^{i-k+1}, \quad i \geq k, \]
so the entire series \( \sum_{i=k}^{\infty} \frac{n_{1} \cdots n_{k-1}q}{n_{i}} \) is bounded by the geometric series \( 1/3 + 1/9 + \cdots \), whose sum is 1/2. Contradiction! So the series is not a rational number.

7. A particle moves from \((0, 0)\) to \((n, n)\) directed by a fair coin. For each head, it moves one step east, and for each tail, one step north. At \((n, y)\), \(y < n\), it stays there if a head comes up; at \((x, n)\), \(x < n\), it stays there if a tail comes up. Let \(k\) be a fixed positive integer. Find the probability that the particle needs exactly \(2n + k\) tosses to reach \((n, n)\).

Consider the infinite integer lattice and assume that having reached a point \((x, n)\) or \((n, y)\), the particle continues moving east and north, following the rules of the game. The required probability \(p_{k}\) is equal to the probability of getting to one of the points \((n, n + k)\), \((n + k, n)\) but without passing through \((n, n + k - 1)\) or \((n + k - 1, n)\). Thus \(p_{k}\) is equal to the probability \(p_{1,k}\) of getting to \((n, n + k)\) via \((n - 1, n + k)\) plus the probability \(p_{2,k}\) of getting to \((n + k, n)\) via \((n + k, n - 1)\). Both probabilities are equal to
\[ \binom{2n + k - 1}{n - 1} 2^{-2n-k} \]
since \(2^{2n+k}\) counts the total number of outcomes, and the binomial \(\binom{2n+k-1}{n-1}\) the number of choices one has to make the requisite number of steps up. Therefore
\[ p_{k} = p_{1,k} + p_{2,k} = \binom{2n + k - 1}{n - 1} 2^{-2n-k+1}. \]

8. Prove that there exists a four-coloring of the set \(M = \{1, 2, \ldots, 1987\}\) such that any arithmetic progression with 10 terms in the set \(M\) is not monochromatic.

Solution. The number of 4-colorings of the set \(M\) is equal to \(4^{1987}\). Let \(A\) be the number of arithmetic progressions in \(M\) with 10 terms. The number of colorings containing a monochromatic arithmetic progression with 10 terms is less than \(4A \cdot 4^{1977}\) since the progression can be assigned any of the 4 colors and since the remaining elements can be any color. So, if \(A < 4^{9}\), then there exist 4-colorings with the required property.

Now we estimate the value of \(A\). If the first term of a 10-term progression is \(k\) and the difference is \(d\), then \(1 \leq k \leq 1978\) and \(d \leq [(1987 - k)/9]\); hence
\[ A = \sum_{k=1}^{1978} \left\lfloor \frac{1987 - k}{9} \right\rfloor < \frac{1986 + 1985 + \cdots + 9}{9} = \frac{1995 \cdot 1978}{18} < 4^{9}. \]