Quadratic reciprocity.

Definitions. An integer $a$ satisfying $(a, m) = 1$ is called a quadratic residue modulo $m$ if there exists a solution to the congruence $x^2 \equiv a \pmod{m}$. Otherwise $a$ is a quadratic nonresidue modulo $m$. Let $p$ be an odd prime. The Legendre symbol $\left( \frac{a}{p} \right)$ is defined as

$$\left( \frac{a}{p} \right) := \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p, \\ -1 & \text{if } a \text{ is a quadratic nonresidue modulo } p, \\ 0 & \text{if } p | a. \end{cases}$$

Basic facts about the Legendre symbol. Let $p$ be an odd prime. Then

1. $a \equiv b \pmod{p} \implies \left( \frac{a}{p} \right) = \left( \frac{b}{p} \right)$,
2. $\left( \frac{a^2}{p} \right) = 1$ unless $p | a$,
3. $\left( \frac{a}{p} \right) \left( \frac{b}{p} \right) = \left( \frac{ab}{p} \right)$,
4. $\left( \frac{-1}{p} \right) = (-1)^{(p-1)/2}$,
5. $\left( \frac{2}{p} \right) = (-1)^{(p^2-1)/8}$.

Theorem [law of quadratic reciprocity]. Let $p$ and $q$ be distinct odd primes. Then

$$\left( \frac{p}{q} \right) \left( \frac{q}{p} \right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}}.$$

Examples.

1. Prove that there are no integers $x$ and $y$ for which

$$x^2 + 3xy - 2y^2 = 122.$$

2. Show that there are no integers $a, b$ for which $2b^2 + 3$ divides $a^2 - 2$.

3. Show that there are infinitely many primes of the form $3k + 1$.

Additional olympiad problems on number theory.

1. Show that the cube roots of three distinct primes cannot be three terms (not necessarily consecutive) of an arithmetic progression.

2. Let $p$ and $q$ be natural numbers such that

$$\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \cdots - \frac{1}{1318} + \frac{1}{1319}.$$

Prove that $p$ is divisible by 1979.
3. Let $s(n)$ denote the sum of all digits of $n$ in decimal notation. Evaluate 

$$s(s(s(4444\ldots4444)))$$.

4. Show that there is no natural number $d$ that makes each of the numbers $2d-1$, $5d-1$, and $13d-1$ a perfect square.

5. Prove that 

$$\cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} = \frac{1}{2}.$$