Polynomials.

**Definitions.** A **polynomial** in \( m \) variables is a function

\[
\sum_{\alpha \in I} c(\alpha)x^\alpha
\]

where \( I \) is a finite subset of \( \mathbb{Z}_+^m \) of multi-indices \( \alpha = (\alpha(1), \ldots, \alpha(m)) \), and the corresponding **monomials** are defined by \( x^\alpha := x_1^{\alpha(1)} \cdots x_m^{\alpha(m)} \). Polynomials are treated as formal expressions in algebra and as functions on \( \mathbb{R}^m \) or \( \mathbb{C}^m \) in analysis. The (nonzero) numbers \( c(\alpha) \) are called **coefficients** of \( p \). The index \( \alpha \) of the monomial occurring in \( p \) that is highest in a chosen monomial order determines the **degree** of \( p \). In the univariate case, the degree is just the biggest power of the variable that occurs in \( p \). In the multivariate setting, various orders are possible, so the same polynomial may have different degrees depending on the order chosen.

Univariate polynomials are by now well understood.

**Fundamental theorem of algebra.** Every nonzero univariate polynomial \( p \) of degree \( n \) with complex coefficients has exactly \( n \) roots \( (a_j)_{j=1}^n \) in \( \mathbb{C} \) and can be factored as

\[
p(z) = a(z - a_1)(z - a_2) \cdots (z - a_n).
\]

**Uniqueness theorem.** If \( p \) and \( q \) are univariate polynomials of degree at most \( n \) and \( p(x_j) = q(x_j) \) for \( j = 1, 2, \ldots, m \) where \( x_1, x_2, \ldots, x_m \) are distinct complex numbers and \( m > n \), then \( p \) and \( q \) are identical.

**Theorem [division algorithm].** If \( f \) and \( g \) are univariate polynomials and \( g \) is not the zero polynomial, then there exist unique polynomials \( q \) and \( r \) such that

\[
f(x) = q(x)g(x) + r(x)
\]

where either \( r \) is the zero polynomial or \( \deg r < \deg g \). The **quotient** \( q \) and the **remainder** \( r \) can be found by synthetic division.

**Bezout’s theorem.** The remainder from the division of a polynomial \( f(x) \) by \( x - a \) is equal to \( f(a) \).

**Theorem [univariate polynomial interpolation].** For any sequence \( (a_j)_{j=1}^n \) of complex numbers and a set of \( n \) distinct points \( (x_j)_{j=1}^n \) from \( \mathbb{C} \), there exists a unique polynomial \( p \) of degree at most \( n - 1 \) such that

\[
p(x_j) = a_j, \quad j = 1, \ldots, n.
\]

**Examples.**

1. Find the remainder when \( x^{81} + x^{49} + x^{25} + x^9 + x \) is divided by \( x^3 - x \).
2. Let \( p \) be a nonconstant polynomial with integral coefficients. If \( n(p) \) is the number of distinct integers \( k \) such that \( (p(k))^2 = 1 \), prove that \( n(p) - \deg(p) \leq 2 \) where \( \deg(p) \) denotes the degree of the polynomial \( p \).
3. Factor \((a + b + c)^3 - (a^3 + b^3 + c^3)\).

4. Find \(a\) if \(a\) and \(b\) are integers such that \(x^2 - x - 1\) is a factor of \(ax^{17} + bx^{16} + 1\).

5. Let \(r \neq 0\) be given. Find the polynomial \(p\) of degree at most \(n\) that satisfies

\[ p(j) = r^j, \quad j = 0, \ldots, n. \]

6. Find the unique polynomial \(p\) of degree \(n\) that satisfies

\[ p(j) = \frac{1}{1 + j}, \quad j = 0, \ldots, n. \]

Hint: consider \((x + 1)p(x) - 1\).

7. A polynomial \(p\) of degree 990 satisfies \(p(k) = F_k\) for \(k = 992, 993, \ldots, 1982\), where \(F_k\) denotes the \(k\)th Fibonacci number. Prove that \(p(1983) = F_{1983} - 1\).