Homework # 3, due Thu, Sep 16th.

1. A positive integer is called evil if the number of digits 1 in its binary expansion is even. For example, 18 = (10010)\(_2\) is evil. Find the sum of the first 1985 evil numbers.

2. Prove that the set of integers of the form \(2^k - 3\), \(k = 2, 3, \ldots\), contains an infinite subset in which every two elements are relatively prime.

3. Show that
\[
\frac{1}{\sqrt{4n}} \leq \left(\frac{1}{2}\right) \left(\frac{3}{4}\right) \cdots \left(\frac{2n-1}{2n}\right) \leq \frac{1}{\sqrt{2n}}.
\]

4. What is the number of subsets of the set \(\{1, \ldots, n\}\) containing exactly one pair of consecutive integers?

5. Consider a regular \(n\)-gon inscribed in a circle of radius 1. What is the product of the lengths of all \(n(n-1)/2\) diagonals of the polygon (which includes its sides)?

6. There are \(n\) seats at a merry-go-round. A boy takes \(n\) rides. Between each ride, he moves clockwise a certain number of places to a new horse. Each time he moves a different number of places. Find all \(n\) for which the boy ends up riding each horse.

7. The function \(f : \mathbb{Z} \to \mathbb{Z}\) satisfies \(f(n) = n - 3\) if \(n \geq 1000\) and \(f(n) = f(f(n + 5))\) if \(n < 1000\). Find \(f(84)\).

8. Prove that the inequality
\[
\sum_{n=1}^{N} \sum_{m=1}^{N} \frac{a_m a_n}{m + n} \geq 0
\]
holds for any real numbers \(a_1, \ldots, a_N\).