1. If $a$ and $b$ are two roots of $x^4 + x^3 - 1 = 0$, prove that $ab$ is a root of $x^6 + x^4 + x^3 - x^2 - 1 = 0$.

2. Consider a $3 \times 3 \times 3$ cube made out of 27 subcubes. The subcubes are connected by doors on their faces, so every subcube has 6 doors. Is it possible to start at the center cube and visit every other cube exactly once?

3. Prove that, for all natural numbers $n$,

$$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1.$$  

4. Suppose every bus ticket has a 6-digit number. A ticket is called lucky if the sum of its first 3 digits equals the sum of its last 3 digits. How many lucky tickets are there?

5. An infinite chessboard contains a positive integer in each square. If the value in each square equals the average of its four neighbors to its south, north, west and east, prove that the values in all squares are equal.

6. The sequence $a_0, a_1, \ldots$ satisfies

$$a_{m+n} + a_{m-n} = \frac{1}{2} (a_{2m} + a_{2n}), \quad m \geq n.$$  

Moreover, $a_1 = 1$. Determine $a_{2010}$.

7. Prove that the points of a right isosceles triangle of side length 1 cannot be colored in four colors so that no two points at distance at least $2 - \sqrt{2}$ receive the same color.

8. A derangement is a permutation with no fixed points. Show that the number of derangement of an $n$-element set is the integer closest to $n!/e$. 