1. Let $x$, $y$ be positive real numbers. Prove that
\[ x^y + y^x > 1. \]

2. Let $x > 0$. Prove that
\[ \sqrt{1 + x} \sqrt{1 + (x + 1)} \sqrt{1 + (x + 2)} \sqrt{1 + \cdots} = x + 1. \]

3. Let $a$, $b$ and $c$ be the side lengths of a triangle. Show that
\[ \frac{3}{2} \leq \frac{a}{b + c} + \frac{b}{a + c} + \frac{c}{a + b} < 2. \]

4. Do there exist integers $a$, $b$, $c$ and $d$ such that the polynomial $p(x) = ax^3 + bx^2 + cx + d$ takes values $p(19) = 1$ and $p(62) = 2$?

5. Let $m$ and $n$ be positive integers. Show that $m^2 + n$ or $n^2 + m$ is not a perfect square.

6. Consider a $6 \times 6$ board tiled by $1 \times 2$ dominos. Prove that the board can be cut into two rectangles, each of which is tiled separately.

7. The Riemann $\zeta$-function is defined as
\[ \zeta(z) := \sum_{n \in \mathbb{N}} \frac{1}{n^z}. \]
Prove that
\[ \sum_{k=0}^{\infty} \frac{1}{(4k + 1)^3} = \frac{\pi^3}{64} + \frac{7}{16} \cdot \zeta(3). \]

8. Show that
\[ \sum_{k=0}^{n} \binom{n}{k} \sin(a + 2k)\theta = 2^n \cos^n \theta \cdot \sin(a + n)\theta. \]