Polynomial equations and symmetric functions.

While algorithms for solving polynomial equations of degree at most 4 exist, there are in general no such algorithms for polynomials of higher degree. A polynomial equation to be solved at an Olympiad is usually solvable by using the Rational Root Theorem (see the earlier handout Rational and Irrational Numbers), symmetry, special forms, and/or symmetric functions.

Here are, for the record, algorithms for solving 3rd and 4th degree equations.

**Algorithm for solving cubic equations.** The general cubic equation
\[ c_0 + c_1 x + c_2 x^2 + c_3 x^3 = 0 \]
can be transformed (by dividing by \( c_3 \) and letting \( z := x + \frac{c_2}{3c_3} \)) into an equation of the form
\[ z^3 + pz + q = 0. \]

To solve this equation, we substitute \( x = u + v \) to obtain
\[ u^3 + v^3 + (u + v)(3uv + p) + q = 0. \]

Note that we are free to restrict \( u \) and \( v \) so that \( uv = -p/3 \). Then \( u^3 \) and \( v^3 \) are the roots of the equation \( z^2 + qz - p^3/27 = 0 \). Solving this equation, we obtain
\[ u^3, v^3 = -\frac{q}{2} \pm \sqrt{R}, \]
where
\[ R := \left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3. \]

Now we may choose cube roots so that
\[ A := \sqrt[3]{-\frac{q}{2} + \sqrt{R}}, \quad B := \sqrt[3]{-\frac{q}{2} - \sqrt{R}}. \]

Then \( A + B \) is a solution. It is easily checked that the other pairs are obtained by rotating \( A \) and \( B \) in the complex plane by angles \( \pm 2\pi/3 \), \( \mp 2\pi/3 \). So, the full set of solutions is
\[ \{A + B, \omega A + \bar{\omega} B, \bar{\omega} A + \omega B\} \quad \text{where} \quad \omega := e^{2\pi i/3}. \]

**Ferrari's method of solving quartic equations.** The general quartic equation is reduced to a cubic equation called the *resolvent*. Write the quartic equation as
\[ x^4 + 2ax^3 + b^2 + 2cx + d = 0. \]

Transpose to obtain
\[ x^4 + 2ax^3 = -b x^2 - 2cx - d \]
and then adding \(2rx^2+(ax+r)^2\) to both sides makes the left-hand side equal to \((x^2+ax+r)^2\). If \(r\) can be chosen to make the right-hand side a perfect square, then it will be easy to find all solutions. The right-hand side,

\[(2r+a^2-b)x^2 + 2(ar-c)x + (r^2-d),\]

is a perfect square if and only if its discriminant is zero. Thus we require

\[2r^3 - br^2 + 2(ac-d)r + (bd - a^2d - c^2) = 0.\]

This is the cubic resolvent.

**Reciprocal or palindromic equations.** If the equation the form \(a_0+a_1x+\cdots+a_nx^n = 0\) and \(a_j = a_{n-j}\) for all \(j = 0, \ldots, n\), it is called *palindromic*. For even \(n\), the transformation \(z := x + \frac{1}{z}\) reduces the equation to a new one of degree \(n/2\). After finding all solutions \(z_j\), the solutions of the original equation are found by solving quadratic equations \(x + \frac{1}{z} = z_j\).

**Examples.**

1. Solve \(x^4 + 2x^3 + 7x^2 + 6x + 8 = 0\).
2. Solve \(x^4 + 2ax^3 + bx^2 + 2ax + 1 = 0\).
3. Solve \(x^3 - 3x + 1 = 0\).
4. Solve \(x^4 - 26x^2 + 72x - 11 = 0\).
5. Solve \(z^4 - 2z^3 + z^2 - a = 0\) and find values of \(a\) for which all roots are real.

**Definitions.** A function of \(n\) variables is *symmetric* if it is invariant under any permutation of its variables. The \(k\)th *elementary symmetric function* is defined by

\[\sigma_k(x_1, \ldots, x_n) := \sum x_{i_1}x_{i_2}\cdots x_{i_k},\]

where the sum is taken over all \(\binom{n}{k}\) choices of the indices \(i_1, i_2, \ldots, i_k\) from the set \(\{1, 2, \ldots, n\}\).

**Symmetric function theorem.** Every symmetric polynomial function of \(x_1, \ldots, x_n\) is a polynomial function of \(\sigma_1, \ldots, \sigma_n\). The same conclusion holds with “polynomial” replaced by “rational function”.

**Theorem.** Let \(x_1, \ldots, x_n\) be the roots of the polynomial equation

\[x^n + c_1x^{n-1} + \cdots + c_n = 0,\]

and let \(\sigma_k\) be the \(k\)th elementary symmetric function of \(x_1, \ldots, x_n\). Then

\[\sigma_k = (-1)^kc_k, \quad k = 1, \ldots, n.\]

**Newton’s formula for power sums.** Let

\[S_p := x_1^p + x_2^p + \cdots + x_n^p, \quad p \in \mathbb{N},\]
where \( x_1, \ldots, x_n \) are the roots of
\[
x^n + c_1x^{n-1} + \cdots + c_n = 0.
\]

Then
\[
\begin{align*}
S_1 + c_1 &= 0 \\
S_2 + c_1S_1 + 2c_2 &= 0 \\
S_3 + c_1S_2 + c_2S_1 + 3c_3 &= 0 \\
&\quad \vdots \\
S_n + c_1S_{n-1} + \cdots + c_{n-1}S_1 + nc_n &= 0 \\
S_p + c_1S_{p-1} + \cdots + c_nS_{p-n} &= 0, \quad p > n.
\end{align*}
\]

Examples.

1. Find all solutions of the system
\[
\begin{align*}
x + y + z &= 0 \\
x^2 + y^2 + z^2 &= 6ab \\
x^3 + y^3 + z^3 &= 3(a^3 + b^3).
\end{align*}
\]

2. If
\[
\begin{align*}
x + y + z &= 1 \\
x^2 + y^2 + z^2 &= 2 \\
x^3 + y^3 + z^3 &= 3,
\end{align*}
\]
determine the value of \( x^4 + y^4 + z^4 \).

3. Let \( G_n := a^n \sin(nA) + b^n \sin(nB) + c^n \sin(nC) \), where \( a, b, c, A, B, C \) are real numbers and \( A + B + C \) is a multiple of \( \pi \). Prove that if \( G_1 = G_2 = 0 \), then \( G_k = 0 \) for all \( k \in \mathbb{N} \).

4. Find a cubic equation whose roots are the cubes of the roots of \( x^3 + ax^2 + bx + c = 0 \).

5. Find all values of the parameter \( a \) such that all roots of the equation
\[
x^6 + 3x^5 + (6 - a)x^4 + (7 - 2a)x^3 + (6 - a)x^2 + 3x + 1 = 0
\]
are real.

6. A student awoke at the end of an algebra class just in time to hear the teacher say, “...and I give you a hint that the roots form an arithmetic progression.” Looking at the board, the student discovered a fifth degree equation to be solved for homework, but he had time to copy only
\[
x^5 - 5x^4 - 35x^3 +
\]
before the teacher erased the blackboard. He was able to find all roots anyway. What are the roots?