Complex numbers.

**Definition.** A *complex number* has the form \( x + iy \) where \( x, y \in \mathbb{R} \) and \( i \) is the *imaginary unit*, which satisfies \( i^2 = -1 \). The set of all complex numbers is denoted by \( \mathbb{C} \). The *real part* \( \text{Re}(z) \) of \( z := x + iy \) is \( x \) and its *imaginary part* \( \text{Im}(z) \) is \( y \). The *complex conjugate* \( \overline{z} \) of \( z \) is \( x - iy \). The *modulus* of \( z \) is \( \sqrt{x^2 + y^2} \).

The sum of two complex numbers, \( x_1 + iy_1 \) and \( x_2 + iy_2 \) is by definition the complex number \( (x_1 + x_2) + i(y_1 + y_2) \) and the product of \( x_1 + iy_1 \) and \( x_2 + iy_2 \) is the number \( (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \).

**Polar representation of a complex number.** Any complex number \( z \) can be written in the form \( ge^{i\phi} \) where \( g \) is the modulus of \( z \) and \( \phi \in [0, 2\pi) \) is its *argument*.

**Euler’s formula.** \( e^{i\phi} = \cos \phi + i \sin \phi \).

**De Moivre’s formula.** \((\cos \phi + i \sin \phi)^n = \cos n\phi + i \sin n\phi \).

**Fundamental theorem of algebra.** If \( n \geq 1 \) and \( a_0, a_1, \ldots, a_n \) are complex numbers with \( a_n \neq 0 \), then there is a complex number \( z \) that satisfies the equation

\[
a_0 + a_1z + \cdots + a_nz^n = 0.
\]

(In fancy terms, the field \( \mathbb{C} \) is algebraically closed.)

**Examples.**

1. Find all \( z \in \mathbb{C} \) that satisfy \( \text{Re}(z) = \text{Im}(z) \) and

\[
|z| + |z + 14| = 28.
\]

2. Find all fifth roots of unity.

3. Find all solutions of the equation \((z - 1)^n = z^n \).

4. Prove the trigonometric identity

\[
\cos^n \theta = \frac{1}{2^n} \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) \cos((n - 2k)\theta).
\]

Hint: recall the *Binomial theorem:*

\[(a + b)^n = \sum_{k=0}^{n} \left( \begin{array}{c} n \\ k \end{array} \right) a^{n-k} b^k.\]

5. Prove the *Multisection formula:* if \( f(x) = \sum_k a_kx^k \), then

\[
\sum_{k \equiv r \pmod{m}} a_kx^k = \frac{1}{m} \sum_{s=0}^{m-1} \varepsilon^{-rs} f(\varepsilon^s x) \quad \text{where} \quad \varepsilon \text{ is a primitive } m\text{th root of 1.}
\]

The latter means that \( \varepsilon^m = 1 \) but \( \varepsilon^j \neq 1 \) for \( j = 1, \ldots, m - 1 \).
6. Evaluate
\[ \sum_{k \equiv 1 \pmod{3}} \binom{n}{k} \quad \text{and} \quad \sum_{k \equiv 2 \pmod{3}} \binom{n}{k}. \]

7. Prove that the number
\[ \sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k} \]

is not divisible by 5 for any integer \( n \geq 0 \).